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NAVAL POSTGRADUATE SCHOOL
Monterey, California

The Modeling of Bistatic Scattering With Moving Platforms

by

Lawrence J. Ziomek

16 October 2002

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

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This report was prepared by:

Lawrence J. Ziomek

LAWRENCE J. ZIOMEK
Professor
Department of Electrical and
Computer Engineering

Reviewed by:

JOHN P. POWERS
Chairman
Department of Electrical and
Computer Engineering

Released by:

DAVID W. NETZER
Associate Provost and
Dean of Research

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13. ABSTRACT *(maximum 200 words)*

The complex frequency response of the ocean is derived for three different bistatic scattering problems. The derivations are based on treating the speed of sound and ambient density of the ocean as constants, and solving for the direct ray path between transmitter and discrete point scatterer, and from discrete point scatterer to receiver. The bistatic scattering problems considered are: 1) no motion, 2) only the discrete point scatterer is in motion, and 3) all three platforms (the transmitter, discrete point scatterer, and receiver) are in motion. The first bistatic scattering problem yields a time-invariant, space-variant complex frequency response while the remaining two bistatic scattering problems yield time-variant, space-variant complex frequency responses. For problems involving motion, the *exact* time-varying ranges between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver are derived, and the *exact* time-varying angles of incidence at the discrete point scatterer, and the *exact* time-varying angles of scatter at the receiver are also derived. The solutions for the exact time-varying ranges are also valid in an inhomogeneous ocean where the speed of sound and ambient density are functions of position since solving for a range represents a problem in mechanics not wave propagation.

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*Department of Electrical and
Computer Engineering*

The Modeling of Bistatic Scattering With Moving Platforms

by

Dr. Lawrence J. Ziomek
Naval Postgraduate School
ECE Department
Code EC/ZM
Monterey, CA 93943

The Modeling of Bistatic Scattering With Moving Platforms

Abstract

The complex frequency response of the ocean is derived for three different bistatic scattering problems. The derivations are based on treating the speed of sound and ambient density of the ocean as constants, and solving for the direct ray path between transmitter and discrete point scatterer, and from discrete point scatterer to receiver. The bistatic scattering problems considered are: 1) no motion, 2) only the discrete point scatterer is in motion, and 3) all three platforms (the transmitter, discrete point scatterer, and receiver) are in motion. The first bistatic scattering problem yields a time-invariant, space-variant complex frequency response while the remaining two bistatic scattering problems yield time-variant, space-variant complex frequency responses. For problems involving motion, the *exact* time-varying ranges between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver are derived, and the *exact* time-varying angles of incidence at the discrete point scatterer, and the *exact* time-varying angles of scatter at the receiver are also derived. The solutions for the exact time-varying ranges are also valid in an inhomogeneous ocean where the speed of sound and ambient density are functions of position since solving for a range represents a problem in mechanics not wave propagation.

1 Introduction

The main purpose of this report is to derive the complex frequency response of the ocean for three different bistatic scattering problems. Since we will be working with *small-amplitude* acoustic signals, a *linear* wave equation accurately describes the propagation of sound between source, discrete point scatterer, and receiver. As a result, we treat the propagation of small-amplitude acoustic signals in the ocean as transmission through a *linear*, *time-variant*, *space-variant filter*. Treating the ocean medium as a *linear* filter is valid because we are trying to solve a *linear* wave equation.

Scatter from a discrete point scatterer is modeled via the *scattering function*, which is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer, and the direction of wave propagation from the scatterer to the receiver. In addition to the scattering function, frequency-dependent attenuation is taken into account in order to model the propagation of sound from transmitter to discrete point scatterer, and from discrete point scatterer to receiver.

The speed of sound and ambient density of the ocean are treated as *constants*. Therefore, sound rays will travel in straight lines. We will only concern ourselves with solving for the direct ray path between transmitter and discrete point scatterer, and from discrete point scatterer to receiver. As a result, the three platforms can be treated as being in an unbounded, homogeneous ocean medium. Sound propagation between the transmitter and the ocean surface and bottom, and from the ocean surface and bottom to the receiver can be handled in the same way as will be developed for the discrete point scatterer.

Section 2.1 is devoted to the first bistatic scattering problem, which involves *no* motion - the transmitter, discrete point scatterer, and receiver are *not* in motion. Most of Section 2.1 follows the analysis presented in [1]. However, what is *new* in Section 2.1 is the derivation of the *exact* solution for the *angles of incidence* at the discrete point scatterer and the *angles of scatter* at the receiver. An example is worked out at the end of Section 2.1 showing how the general bistatic scattering results reduce for a *monostatic* (*backscatter*) scattering geometry for the no motion case.

Section 2.2 is devoted to the second bistatic scattering problem when only the discrete point scatterer is in motion. The initial analysis in Section 2.2 follows that presented in [2], with the exception that motion is now allowed to start at an arbitrary time instant t_m seconds as opposed to

zero seconds as was done in [2]. In addition, two *new* major results are presented in Section 2.2: 1) the *exact time-varying ranges* between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver are derived, and 2) the *exact time-varying angles of incidence* at the discrete point scatterer, and the *exact time-varying angles of scatter* at the receiver are also derived. An example is worked out at the end of Section 2.2 showing how the general bistatic scattering results reduce for a *monostatic* (*backscatter*) scattering geometry for the case when only the discrete point scatterer is in motion.

Section 2.3 is devoted to the third bistatic scattering problem when all three platforms are in motion. The initial analysis in Section 2.3 follows that presented in [3], with the exception that motion is now allowed to start at an arbitrary time instant t_m seconds as opposed to zero seconds as was done in [3]. In addition, two *new* major results are presented in Section 2.3: 1) the *exact time-varying ranges* between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver are derived, and 2) the *exact time-varying angles of incidence* at the discrete point scatterer, and the *exact time-varying angles of scatter* at the receiver are also derived. Three examples are worked out at the end of Section 2.3. The first example shows that the exact results derived in Section 2.3 reduce to the exact results derived in Sections 2.1 and 2.2 when appropriate values are used for the various parameters. This is a very important example because it validates the correctness of the general solution derived in Section 2.3. The second example shows how the general bistatic scattering results reduce for a *monostatic* (*backscatter*) scattering geometry for the case when all three platforms are in motion. The third example shows how the general bistatic scattering results can be applied to a synthetic aperture sonar (SAS) system trying to image a nonmoving target on the ocean bottom without having to make several simplifying assumptions as is done, for example, in [8].

Finally, note that for problems involving motion, the solutions for the exact time-varying ranges between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver derived in this report are also valid in an *inhomogeneous ocean* where the speed of sound and ambient density are functions of position since solving for a range represents a problem in mechanics not wave propagation. However, travel times and angles of incidence and scatter are different in an inhomogeneous ocean compared to a homogeneous ocean because of the complicated trajectories of sound rays in an inhomogeneous ocean.

2 Bistatic Scattering

2.1 No Motion

In this section we will analyze the simple bistatic scattering problem shown in Fig. 2.1-1. The transmitter, discrete point scatterer, and receiver are *not* in motion. No motion corresponds to a *time-invariant* problem. As mentioned in the Introduction, the three platforms will be treated as being in an unbounded, homogeneous ocean medium. Although the propagation of sound between the source and discrete point scatterer, and between the discrete point scatterer and receiver can be treated as transmission through linear, *time-invariant*, *space-invariant* filters; the overall solution for this bistatic scattering problem corresponds to transmission through a linear, *time-invariant*, *space-variant* filter. The presence of a discrete point scatterer in an unbounded, homogeneous fluid medium (i.e., a fluid medium with constant speed of sound and ambient density) causes the medium to be space-variant.

Let the source distribution $x_M(t,\mathbf{r})$ at time t and position $\mathbf{r}=(x,y,z)$ be a *motionless*, *time-harmonic*, *point source* with units of inverse seconds, that is, let

$$x_M(t,\mathbf{r}) = S_0\delta(\mathbf{r}-\mathbf{r}_0)\exp(+j2\pi ft), \tag{2.1-1}$$

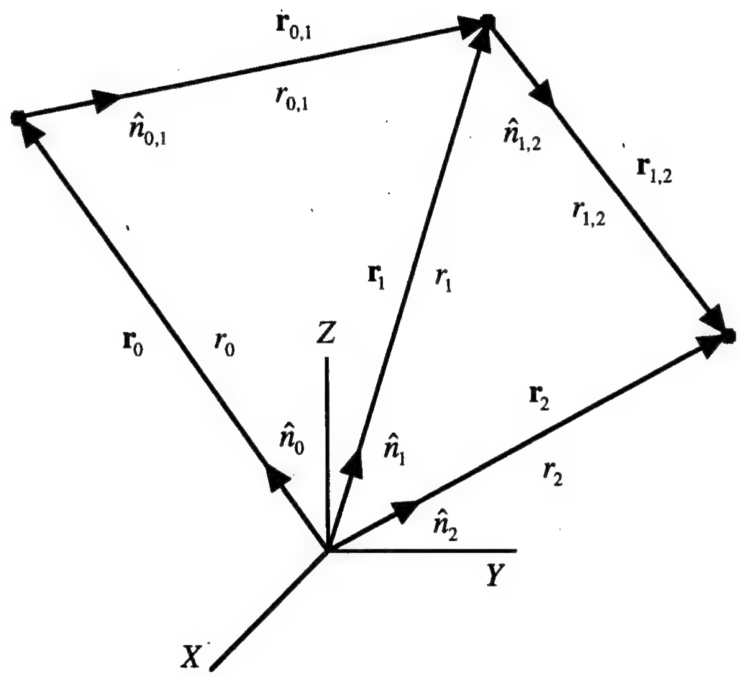

Figure 2.1-1. Bistatic scattering geometry. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. None of the platforms are in motion.

where S_0 is the *source strength* in cubic meters per second, the impulse function $\delta(\mathbf{r}-\mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0=(x_0,y_0,z_0)$, and f is frequency in hertz. The sound source has been turned on forever, that is, since $t=-\infty$. The propagation of sound between the source and discrete point scatterer can be modeled as transmission through a linear, *time-invariant*, *space-invariant*, filter. Therefore, the acoustic field (velocity potential) incident upon the discrete point scatterer at $\mathbf{r}_1=(x_1,y_1,z_1)$ is given by [4]

$$y_M(t,\mathbf{r}_1)=y_{f,M}(\mathbf{r}_1)\exp(+j2\pi ft), \tag{2.1-2}$$

where

$$y_{f,M}(\mathbf{r}_1)=S_0H_M(f,\mathbf{r}_1-\mathbf{r}_0), \tag{2.1-3}$$

$$H_M(f,\mathbf{r}_1-\mathbf{r}_0)=-\frac{\exp\left(-jk\left|\mathbf{r}_1-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1-\mathbf{r}_0\right|} \tag{2.1-4}$$

is the *time-invariant*, *space-invariant*, *complex frequency response* of the ocean at frequency f hertz,

$$k=2\pi f/c=2\pi/\lambda \tag{2.1-5}$$

is the *wavenumber* with units of radians per meter, c is the *constant* speed of sound of the homogeneous ocean medium in meters per second, and λ is the *wavelength* in meters.

As can be seen from Fig. 2.1-1, the position vector from the point source to the discrete point scatterer is given by

$$\mathbf{r}_{0,1}=\mathbf{r}_1-\mathbf{r}_0. \tag{2.1-6}$$

Therefore, (2.1-3) and (2.1-4) can be rewritten as

$$y_{f,M}(\mathbf{r}_1)=S_0H_M(f,\mathbf{r}_{0,1}) \tag{2.1-7}$$

and

$$H_M(f,\mathbf{r}_{0,1})=-\frac{\exp\left(-jk\left|\mathbf{r}_{0,1}\right|\right)}{4\pi\left|\mathbf{r}_{0,1}\right|}, \tag{2.1-8}$$

respectively.

In order to compute the acoustic signal incident upon the receiver, we treat the discrete point scatterer as another *motionless*, *time-harmonic*, *point source* with units of inverse seconds, that is, let [see (2.1-1) and Fig. 2.1-2]

$$x'_M(t,\mathbf{r})=S'_0\delta(\mathbf{r}-\mathbf{r}_1)\exp(+j2\pi ft), \tag{2.1-9}$$

where S'_0 is the *source strength* in cubic meters per second, and the impulse function $\delta(\mathbf{r}-\mathbf{r}_1)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_1=(x_1,y_1,z_1)$. The source strength S'_0 is given by

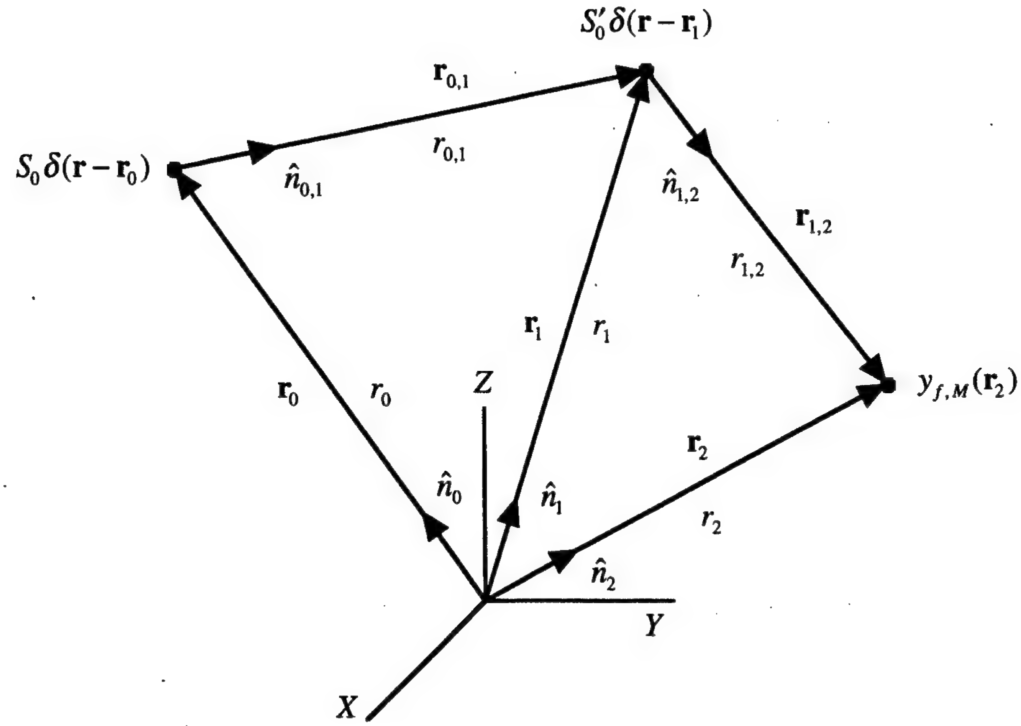

Figure 2.1-2. Bistatic scattering geometry. Both the transmitter at point 0, $P_0(\mathbf{r}_0)$, and the discrete point scatterer at point 1, $P_1(\mathbf{r}_1)$, are time-harmonic point sources. None of the platforms are in motion.

$$S_0' = y_{f,M}(\mathbf{r}_1) g_1\left(f, \hat{n}_{0,1}, \hat{n}_{1,2}\right), \tag{2.1-10}$$

where $y_{f,M}(\mathbf{r}_1)$ is given by (2.1-3) and is the spatial-dependent part of the time-harmonic velocity potential at $\mathbf{r}_1$ with units of squared-meters per second, $g_1\left(f, \hat{n}_{0,1}, \hat{n}_{1,2}\right)$ is the *scattering function* of the discrete point scatterer with units of meters, and $\hat{n}_{0,1}$ and $\hat{n}_{1,2}$ are the dimensionless unit vectors in the directions measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively. The unit of meters for the scattering function represents an *effective scattering length* that may be larger or smaller than the actual length of the scatterer. The scattering function is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer ($\hat{n}_{0,1}$), and the direction of wave propagation from the scatterer to the receiver ($\hat{n}_{1,2}$) (see either Fig. 2.1-1 or Fig. 2.1-2) [5]. Later in this section, we will show how to express the scattering function as not only a function of frequency, but also as a function of the *angles of incidence and scatter* instead of unit vectors. The use of unit vectors is meant as a shorthand notation.

The propagation of sound between the discrete point scatterer and receiver can also be modeled as transmission through a linear, *time-invariant, space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$ due to a point source at $\mathbf{r}_1 = (x_1, y_1, z_1)$ is given by [4]

$$y_M(t, \mathbf{r}_2) = y_{f,M}(\mathbf{r}_2) \exp(+j2\pi f t), \tag{2.1-11}$$

where

$$y_{f,M}(\mathbf{r}_2) = S_0' H_M(f, \mathbf{r}_2 - \mathbf{r}_1) \tag{2.1-12}$$

and

$$H_M(f, \mathbf{r}_2 - \mathbf{r}_1) = -\frac{\exp\left(-jk\left|\mathbf{r}_2 - \mathbf{r}_1\right|\right)}{4\pi\left|\mathbf{r}_2 - \mathbf{r}_1\right|}. \tag{2.1-13}$$

As can be seen from either Fig. 2.1-1 or Fig. 2.1-2, the position vector from the discrete point scatterer to the receiver is given by

$$\mathbf{r}_{1,2} = \mathbf{r}_2 - \mathbf{r}_1. \tag{2.1-14}$$

Therefore, (2.1-12) and (2.1-13) can be rewritten as

$$y_{f,M}(\mathbf{r}_2) = S_0' H_M(f, \mathbf{r}_{1,2}) \tag{2.1-15}$$

and

$$H_M(f, \mathbf{r}_{1,2}) = -\frac{\exp\left(-jk\left|\mathbf{r}_{1,2}\right|\right)}{4\pi\left|\mathbf{r}_{1,2}\right|}, \tag{2.1-16}$$

respectively.

Let us now begin the process of obtaining a final expression for the time-harmonic velocity potential incident upon the receiver. Substituting (2.1-10) and (2.1-7) into (2.1-15) yields

$$y_{f,M}(\mathbf{r}_2) = S_0 H_M(f,\mathbf{r}_{0,1}) g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) H_M(f,\mathbf{r}_{1,2}), \tag{2.1-17}$$

or, equivalently,

$$y_{f,M}(\mathbf{r}_2) = S_0 H_M\left(f,\mathbf{r}_2 \,|\, \mathbf{r}_0\right), \tag{2.1-18}$$

where

$$\begin{aligned} H_M\left(f,\mathbf{r}_2 \,|\, \mathbf{r}_0\right) &= H_M(f,\mathbf{r}_{0,1}) g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) H_M(f,\mathbf{r}_{1,2}) \\ &= g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) \frac{\exp\left[-jk\left(\left|\mathbf{r}_{0,1}\right| + \left|\mathbf{r}_{1,2}\right|\right)\right]}{16\pi^2 \left|\mathbf{r}_{0,1}\right| \left|\mathbf{r}_{1,2}\right|} \\ &= g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) \frac{\exp\left[-jk\left(\left|\mathbf{r}_1 - \mathbf{r}_0\right| + \left|\mathbf{r}_2 - \mathbf{r}_1\right|\right)\right]}{16\pi^2 \left|\mathbf{r}_1 - \mathbf{r}_0\right| \left|\mathbf{r}_2 - \mathbf{r}_1\right|}. \end{aligned} \tag{2.1-19}$$

Note that if the bistatic scattering problem shown in Fig. 2.1-1 corresponded to transmission through a space-invariant filter, then the complex frequency response given by (2.1-19) would be a function of the vector spatial difference $\mathbf{r}_2 - \mathbf{r}_0$, which it is *not*. Also note that in order to model the effects of frequency-dependent attenuation, simply replace the real wavenumber k given by (2.1-5) with the following *complex wavenumber* K:

$$\boxed{K = k - j\alpha(f),} \tag{2.1-20}$$

where $\alpha(f)$ is the *real, frequency-dependent, attenuation coefficient* in nepers per meter. In addition to being real quantities, both k and $\alpha(f)$ are *positive.*

With the use of (2.1-5), (2.1-11), (2.1-18), and (2.1-19); and by replacing the real wavenumber k in (2.1-19) with the complex wavenumber K given by (2.1-20), we can *summarize* our results as follows: for the bistatic scattering problem shown in Fig. 2.1-1, the time-harmonic velocity potential in squared-meters per second incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$, due to a time-harmonic point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a discrete point scatterer at $\mathbf{r}_1 = (x_1, y_1, z_1)$, is given by

$$\boxed{y_M(t,\mathbf{r}_2) = S_0 H_M\left(f,\mathbf{r}_2 \,|\, \mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t \geq \tau,} \tag{2.1-21}$$

where S_0 is the source strength in cubic meters per second,

$$\boxed{H_M\left(f,\mathbf{r}_2 \,|\, \mathbf{r}_0\right) = g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) \frac{\exp\left[-\alpha(f)(r_{0,1} + r_{1,2})\right]}{16\pi^2 r_{0,1} r_{1,2}} \exp(-j2\pi f\tau)} \tag{2.1-22}$$

is the time-invariant, space-variant, complex frequency response of the ocean at frequency f hertz, $g_1(f,\hat{n}_{0,1},\hat{n}_{1,2})$ is the scattering function of the discrete point scatterer in meters [see (2.1-46)],

$$\hat{n}_{0,1} = \frac{\mathbf{r}_{0,1}}{|\mathbf{r}_{0,1}|} = \frac{\mathbf{r}_1 - \mathbf{r}_0}{|\mathbf{r}_1 - \mathbf{r}_0|} \tag{2.1-23}$$

and

$$\hat{n}_{1,2} = \frac{\mathbf{r}_{1,2}}{|\mathbf{r}_{1,2}|} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{2.1-24}$$

are the dimensionless unit vectors in the directions measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively, $\alpha(f)$ is the real, frequency-dependent, attenuation coefficient in nepers per meter,

$$\boxed{\tau = \frac{r_{0,1} + r_{1,2}}{c}} \tag{2.1-25}$$

is the time delay in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver), and

$$r_{0,1} = |\mathbf{r}_{0,1}| = |\mathbf{r}_1 - \mathbf{r}_0| \tag{2.1-26}$$

and

$$r_{1,2} = |\mathbf{r}_{1,2}| = |\mathbf{r}_2 - \mathbf{r}_1| \tag{2.1-27}$$

are the distances (ranges) in meters measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively.

Equations (2.1-23) and (2.1-24) represent one way to compute the unit vectors $\hat{n}_{0,1}$ and $\hat{n}_{1,2}$. Unit vectors can also be computed using direction cosines. For example, the dimensionless unit vector $\hat{n}_{0,1}$ can also be expressed as follows:

$$\hat{n}_{0,1} = u_{0,1}\hat{x} + v_{0,1}\hat{y} + w_{0,1}\hat{z}, \tag{2.1-28}$$

where

$$u_{0,1} = \sin\theta_{0,1}\cos\psi_{0,1}, \tag{2.1-29}$$

$$v_{0,1} = \sin\theta_{0,1}\sin\psi_{0,1}, \tag{2.1-30}$$

and

$$w_{0,1} = \cos\theta_{0,1} \tag{2.1-31}$$

are dimensionless direction cosines with respect to the X, Y, and Z axes, respectively, and $\theta_{0,1}$ and $\psi_{0,1}$ are spherical angles defined in Fig. 2.1-3. Equating (2.1-23) and (2.1-28) yields

$$u_{0,1} = \frac{x_1 - x_0}{|\mathbf{r}_1 - \mathbf{r}_0|}, \tag{2.1-32}$$

$$v_{0,1} = \frac{y_1 - y_0}{|\mathbf{r}_1 - \mathbf{r}_0|}, \tag{2.1-33}$$

and

$$w_{0,1} = \frac{z_1 - z_0}{|\mathbf{r}_1 - \mathbf{r}_0|}. \tag{2.1-34}$$

Therefore, with the use of (2.1-29) through (2.1-31), and (2.1-32) through (2.1-34), the spherical angles $\theta_{0,1}$ and $\psi_{0,1}$ can be computed as follows:

$$\boxed{\theta_{0,1} = \cos^{-1} w_{0,1} = \cos^{-1}\left(\frac{z_1 - z_0}{|\mathbf{r}_1 - \mathbf{r}_0|}\right)} \tag{2.1-35}$$

and

$$\boxed{\psi_{0,1} = \tan^{-1}\left(\frac{v_{0,1}}{u_{0,1}}\right) = \tan^{-1}\left(\frac{y_1 - y_0}{x_1 - x_0}\right).} \tag{2.1-36}$$

Equations (2.1-35) and (2.1-36) are the *angles of incidence* at the discrete point scatterer.

Similarly, the dimensionless unit vector $\hat{n}_{1,2}$ can also be expressed as follows:

$$\hat{n}_{1,2} = u_{1,2}\hat{x} + v_{1,2}\hat{y} + w_{1,2}\hat{z}, \tag{2.1-37}$$

where

$$u_{1,2} = \sin\theta_{1,2}\cos\psi_{1,2}, \tag{2.1-38}$$

$$v_{1,2} = \sin\theta_{1,2}\sin\psi_{1,2},, \tag{2.1-39}$$

and

$$w_{1,2} = \cos\theta_{1,2} \tag{2.1-40}$$

are dimensionless direction cosines with respect to the X, Y, and Z axes, respectively, and $\theta_{1,2}$ and $\psi_{1,2}$ are spherical angles defined in Fig. 2.1-4. Equating (2.1-24) and (2.1-37) yields

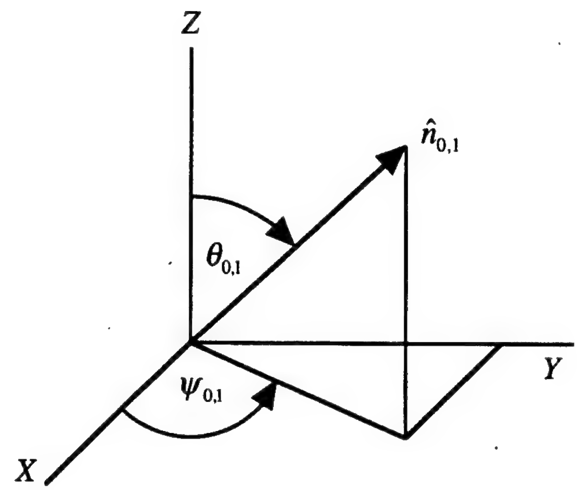

Figure 2.1-3. The dimensionless unit vector $\hat{n}_{0,1}$ measured in the direction from the source to the discrete point scatterer, and associated spherical angles $\theta_{0,1}$ and $\psi_{0,1}$.

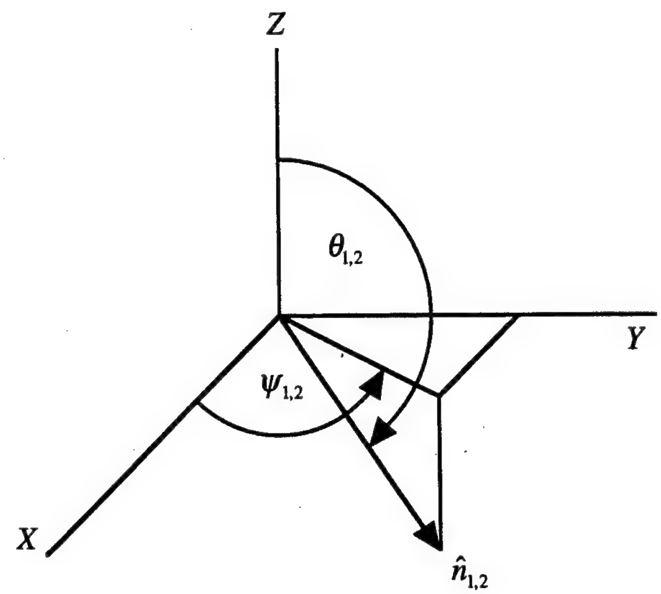

Figure 2.1-4. The dimensionless unit vector $\hat{n}_{1,2}$ measured in the direction from the discrete point scatterer to the receiver, and associated spherical angles $\theta_{1,2}$ and $\psi_{1,2}$.

$$u_{1,2} = \frac{x_2 - x_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \tag{2.1-41}$$

$$v_{1,2} = \frac{y_2 - y_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \tag{2.1-42}$$

and

$$w_{1,2} = \frac{z_2 - z_1}{|\mathbf{r}_2 - \mathbf{r}_1|}. \tag{2.1-43}$$

Therefore, with the use of (2.1-38) through (2.1-40), and (2.1-41) through (2.1-43), the spherical angles $\theta_{1,2}$ and $\psi_{1,2}$ can be computed as follows:

$$\boxed{\theta_{1,2} = \cos^{-1} w_{1,2} = \cos^{-1}\left(\frac{z_2 - z_1}{|\mathbf{r}_2 - \mathbf{r}_1|}\right)} \tag{2.1-44}$$

and

$$\boxed{\psi_{1,2} = \tan^{-1}\left(\frac{v_{1,2}}{u_{1,2}}\right) = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right).} \tag{2.1-45}$$

Equations (2.1-44) and (2.1-45) are the *angles of scatter* at the receiver.

The result of the above analysis is that the scattering function can also be expressed as a function of two sets of angles – the angles of incidence $(\theta_{0,1}, \psi_{0,1})$ given by (2.1-35) and (2.1-36), and the angles of scatter $(\theta_{1,2}, \psi_{1,2})$ given by (2.1-44) and (2.1-45). Therefore,

$$\boxed{g_1(f, \hat{n}_{0,1}, \hat{n}_{1,2}) \equiv g_1(f, \theta_{0,1}, \psi_{0,1}, \theta_{1,2}, \psi_{1,2}).} \tag{2.1-46}$$

Let us next relate the scattering function, the differential scattering cross section, and target strength of the discrete point scatterer. The *target strength* (TS) is defined as follows [6]:

$$\boxed{\mathrm{TS} \triangleq 10\log_{10}\left[\frac{\sigma_d(f, \hat{n}_{0,1}, \hat{n}_{1,2})}{A_{\mathrm{ref}}}\right] \mathrm{dB\ re}\ A_{\mathrm{ref}},} \tag{2.1-47}$$

where [5, 6]

$$\boxed{\sigma_d\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) \triangleq \lim_{r_{1,2}\to\infty}\left[\frac{r_{1,2}^2\, I_{avg_s}(\mathbf{r}_2)}{I_{avg_i}(\mathbf{r}_1)}\right] = \frac{\left|g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)\right|^2}{(4\pi)^2}} \tag{2.1-48}$$

is the *differential scattering cross section* with units of squared meters, $I_{avg_i}(\mathbf{r}_1)$ and $I_{avg_s}(\mathbf{r}_2)$ are the *time-average*, *incident* and *scattered intensities*, respectively, with units of watts per squared meter, $g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)$ is the scattering function of the discrete point scatterer with units of meters, and A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to $1\ \text{m}^2$.

Example 2.1-1 Monostatic Scattering Geometry

For a *monostatic* (*backscatter*) scattering geometry, both the transmitter and receiver are located at the same position, that is,

$$\mathbf{r}_2 = \mathbf{r}_0 . \tag{2.1-49}$$

Substituting (2.1-49) into (2.1-14) yields

$$\mathbf{r}_{1,2} = -\mathbf{r}_{0,1}, \tag{2.1-50}$$

where $\mathbf{r}_{0,1}$ is given by (2.1-6). Therefore,

$$r_{1,2} = r_{0,1} \tag{2.1-51}$$

and

$$\hat{n}_{1,2} = -\hat{n}_{0,1} . \tag{2.1-52}$$

With the use of (2.1-51) and (2.1-52), the time-invariant, space-variant, complex frequency response of the ocean given by (2.1-22) reduces to

$$H_M\left(f,\mathbf{r}_2\,|\,\mathbf{r}_0\right) = g_1\left(f,\hat{n}_{0,1},-\hat{n}_{0,1}\right)\frac{\exp\left[-2\alpha(f)r_{0,1}\right]}{(4\pi r_{0,1})^2}\exp(-j2\pi f\tau), \tag{2.1-53}$$

where

$$g_1\left(f,\hat{n}_{0,1},-\hat{n}_{0,1}\right) \equiv g_1\left(f,\theta_{0,1},\psi_{0,1},\pi-\theta_{0,1},\pi+\psi_{0,1}\right), \tag{2.1-54}$$

$(\theta_{0,1},\psi_{0,1})$ are the angles of incidence given by (2.1-35) and (2.1-36), $\theta_{1,2} = \pi - \theta_{0,1}$ and $\psi_{1,2} = \pi + \psi_{0,1}$ are the angles of scatter,

$$\tau = \frac{2r_{0,1}}{c} \tag{2.1-55}$$

is the time delay in seconds, and $r_{0,1}$ is given by (2.1-26).

2.2 Discrete Point Scatterer In Motion

In this section we will analyze the bistatic scattering problem shown in Fig. 2.2-1. The transmitter and receiver are *not* in motion. Only the discrete point scatterer is in motion. Motion corresponds to a *time-variant* problem. As mentioned in the Introduction, the three platforms will be treated as being in an unbounded, homogeneous ocean medium. Although the propagation of sound between the source and discrete point scatterer, and between the discrete point scatterer and receiver can be treated as transmission through linear, *time-variant, space-invariant* filters; the overall solution for this bistatic scattering problem corresponds to transmission through a linear, *time-variant, space-variant* filter. The presence of a discrete point scatterer in an unbounded, homogeneous fluid medium (i.e., a fluid medium with constant speed of sound and ambient density) causes the medium to be space-variant.

Let the source distribution $x_M(t,\mathbf{r})$ at time t and position $\mathbf{r}=(x,y,z)$ be a *motionless, time-harmonic, point source* with units of inverse seconds, that is, let

$$x_M(t,\mathbf{r}) = S_0\delta(\mathbf{r}-\mathbf{r}_0)\exp(+j2\pi ft), \tag{2.2-1}$$

where S_0 is the *source strength* in cubic meters per second, the impulse function $\delta(\mathbf{r}-\mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0=(x_0,y_0,z_0)$, and f is frequency in hertz. The sound source has been turned on forever, that is, since $t=-\infty$. The *velocity vector* of the discrete point scatterer, $\mathbf{V}_1$, is given by

$$\mathbf{V}_1 = V_1\hat{n}_{V_1}, \tag{2.2-2}$$

where V_1 is the *speed* in meters per second and $\hat{n}_{V_1}$ is the dimensionless unit vector in the direction of $\mathbf{V}_1$. The velocity vector given by (2.2-2) is *constant*, that is, both the speed and direction are *constants* - there is *no* acceleration. Motion begins at time $t=t_m$ seconds. We will model the propagation of sound from the time motion begins.

Since the discrete point scatterer is now in motion, the position vector from the origin to the discrete point scatterer, denoted by $\mathcal{R}_1(t)$, is a function of time given by

$$\mathcal{R}_1(t) = \mathbf{r}_1 + \Delta t\,\mathbf{V}_1, \qquad t \ge t_m, \tag{2.2-3}$$

where $\mathbf{r}_1=(x_1,y_1,z_1)$ is the position vector from the origin to the discrete point scatterer when motion begins (see Fig. 2.2-1), and

$$\Delta t = t - t_m, \qquad t \ge t_m. \tag{2.2-4}$$

Note that $\mathcal{R}_1(t_m)=\mathbf{r}_1$.

When the transmitted acoustic field is first incident upon the discrete point scatterer at some time t' seconds where $t'>t_m$, the position vector from the origin to the discrete point scatterer is given by [see (2.2-3) and Fig. 2.2-2]

$$\mathbf{r}_1' = \mathcal{R}_1(t') = \mathbf{r}_1 + \Delta t'\,\mathbf{V}_1, \qquad t' > t_m, \tag{2.2-5}$$

where

$$\Delta t' = t' - t_m, \qquad t' > t_m. \tag{2.2-6}$$

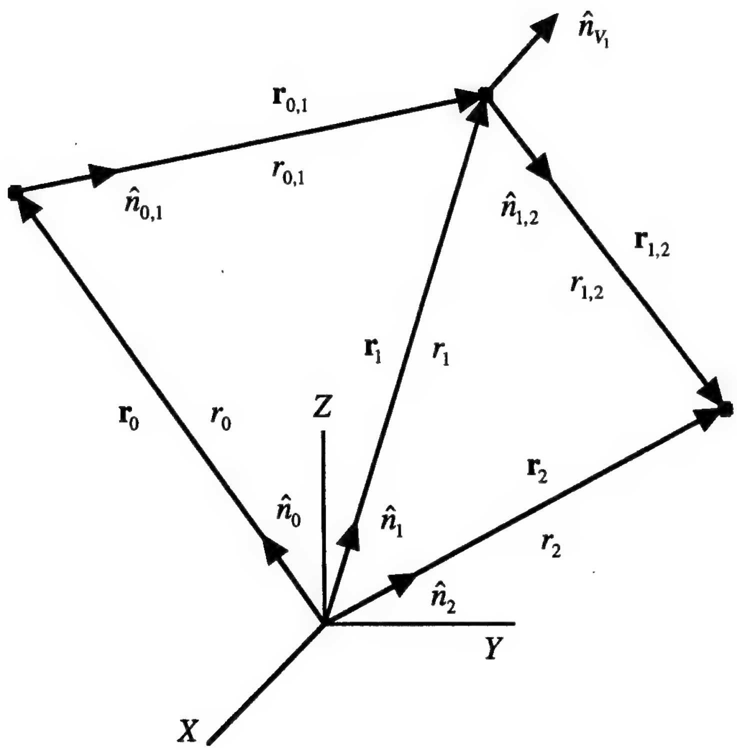

Figure 2.2-1. Bistatic scattering geometry when motion begins at time $t = t_m$ seconds. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. Only the discrete point scatterer is in motion.

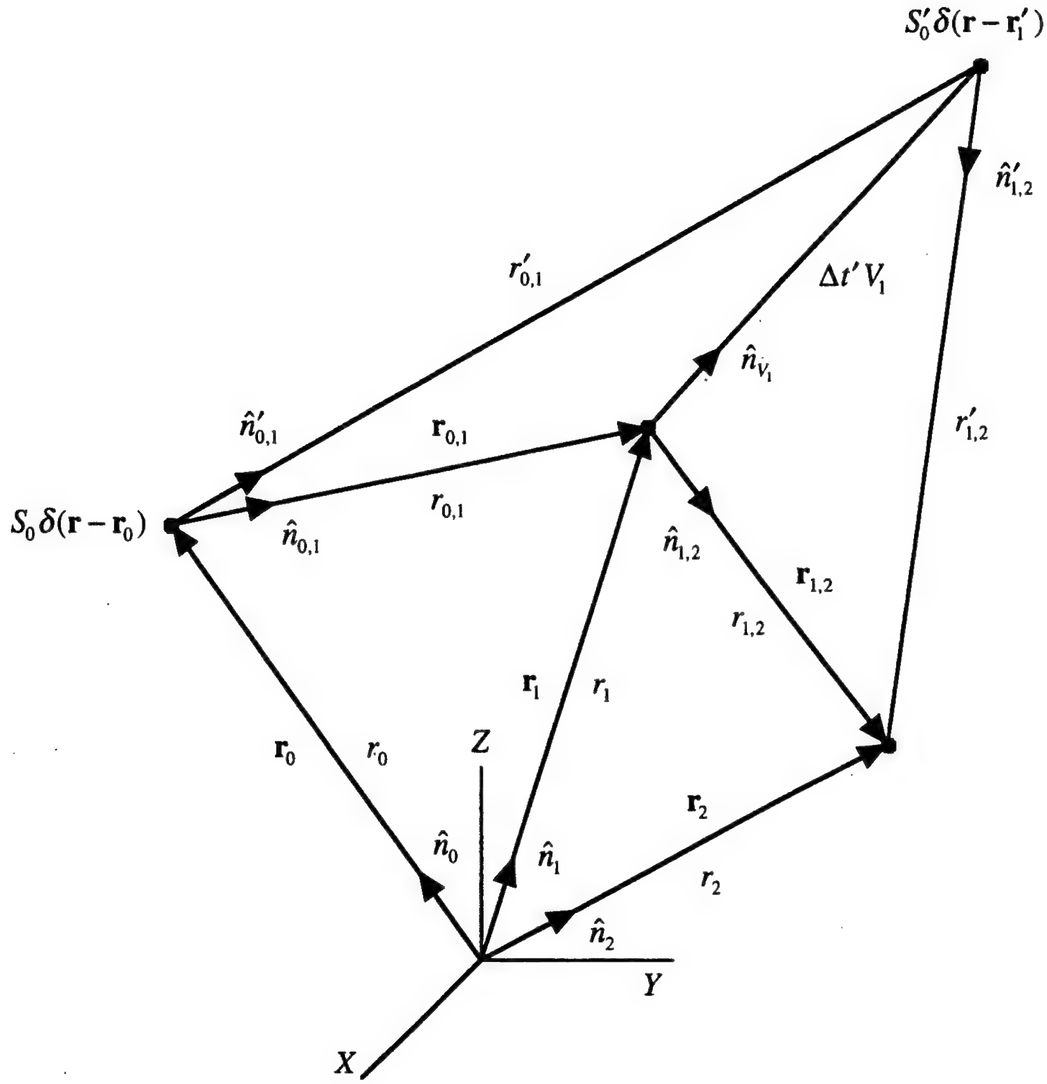

Figure 2.2-2. Bistatic scattering geometry when the transmitted acoustic field is first incident upon the discrete point scatterer at time t' seconds. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. Only the discrete point scatterer is in motion.

The propagation of sound between the source and discrete point scatterer can be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the discrete point scatterer at time t' and position $\mathbf{r}_1' = (x_1', y_1', z_1')$ is given by [7]

$$y_M(t', \mathbf{r}_1') = S_0 H_M\left(t', \mathbf{r}_1' - \mathbf{r}_0 \,|\, f\right) \exp(+j2\pi f t'), \qquad t' > t_m, \tag{2.2-7}$$

where

$$H_M\left(t', \mathbf{r}_1' - \mathbf{r}_0 \,|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_1' - \mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1' - \mathbf{r}_0\right|} \tag{2.2-8}$$

is the *time-variant, space-invariant, complex frequency response* of the ocean at frequency f hertz, and k is the wavenumber in radians per meter given by (2.1-5) and is repeated below for convenience:

$$k = 2\pi f / c = 2\pi / \lambda . \tag{2.1-5}$$

By referring to Fig. 2.2-2, we can express the position vector from the point source to the discrete point scatterer at time t' as

$$\mathbf{r}_{0,1}' = \mathbf{r}_1' - \mathbf{r}_0, \tag{2.2-9}$$

and upon substituting (2.2-5) into (2.2-9), we obtain

$$\mathbf{r}_{0,1}' = r_{0,1}' \hat{n}_{0,1}' = \mathbf{r}_{0,1} + \Delta t' \mathbf{V}_1, \tag{2.2-10}$$

where $r_{0,1}' = \left|\mathbf{r}_{0,1}'\right|$, $\hat{n}_{0,1}'$ is the dimensionless unit vector in the direction of $\mathbf{r}_{0,1}'$, $\mathbf{r}_{0,1}$ is given by (2.1-6) and is repeated below for convenience (also see Fig. 2.2-2),

$$\mathbf{r}_{0,1} = \mathbf{r}_1 - \mathbf{r}_0, \tag{2.1-6}$$

and $\Delta t'$ is given by (2.2-6). Therefore, (2.2-7) and (2.2-8) can be rewritten as

$$y_M(t', \mathbf{r}_1') = S_0 H_M\left(t', \mathbf{r}_{0,1}' \middle| f\right) \exp(+j2\pi f t'), \qquad t' > t_m, \tag{2.2-11}$$

and

$$H_M\left(t', \mathbf{r}_{0,1}' \middle| f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_{0,1}'\right|\right)}{4\pi\left|\mathbf{r}_{0,1}'\right|}. \tag{2.2-12}$$

In order to compute the acoustic signal incident upon the receiver, we treat the discrete point scatterer at time $t \geq t' > t_m$ and position $\mathbf{r}_1'$ as another *motionless*, *time-harmonic*, *point source* with units of inverse seconds, that is, let [see (2.2-1) and Fig. 2.2-2]

$$x_M'(t, \mathbf{r}) = S_0' \delta(\mathbf{r} - \mathbf{r}_1') \exp(+j2\pi f t), \tag{2.2-13}$$

where S_0' is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r}-\mathbf{r}_1')$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_1'=(x_1',y_1',z_1')$. The source strength S_0' is given by

$$S_0' = S_0 H_M\left(t',\mathbf{r}_{0,1}'\middle|f\right)g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right), \tag{2.2-14}$$

where $g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right)$ is the *scattering function* of the discrete point scatterer with units of meters as discussed in Section 2.1. Since the discrete point scatterer is now in motion and the transmitted acoustic field is first incident upon the discrete point scatterer at time t', the direction of wave propagation from the source to the scatterer is given by the dimensionless unit vector $\hat{n}_{0,1}'$ (see Fig. 2.2-2). Similarly, since the discrete point scatterer is being treated as another point source at time t' and position $\mathbf{r}_1'$, the direction of wave propagation from the scatterer to the receiver is given by the dimensionless unit vector $\hat{n}_{1,2}'$ (see Fig. 2.2-2). Later in this section, we will show that the scattering function is also a function of time because the unit vectors are actually *time-varying*. We will then show how to express the scattering function as a function of frequency and *time-varying angles of incidence and scatter* instead of time-varying unit vectors. The use of unit vectors is meant as a shorthand notation.

The propagation of sound between the discrete point scatterer and receiver can also be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. The scattered acoustic field is first incident upon the receiver at some time t seconds where $t > t' > t_m$. Therefore, the acoustic field (velocity potential) incident upon the receiver at time t and position $\mathbf{r}_2=(x_2,y_2,z_2)$ due to a point source at $\mathbf{r}_1'=(x_1',y_1',z_1')$ is given by [7]

$$y_M(t,\mathbf{r}_2) = S_0' H_M\left(t,\mathbf{r}_2-\mathbf{r}_1'\middle|f\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.2-15}$$

where

$$H_M\left(t,\mathbf{r}_2-\mathbf{r}_1'\middle|f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_2-\mathbf{r}_1'\right|\right)}{4\pi\left|\mathbf{r}_2-\mathbf{r}_1'\right|}. \tag{2.2-16}$$

By referring to Fig. 2.2-2, we can express the position vector from the discrete point scatterer to the receiver at time $t' > t_m$ as

$$\mathbf{r}_{1,2}' = \mathbf{r}_2 - \mathbf{r}_1', \tag{2.2-17}$$

and upon substituting (2.2-5) into (2.2-17), we obtain

$$\mathbf{r}_{1,2}' = r_{1,2}'\hat{n}_{1,2}' = \mathbf{r}_{1,2} - \Delta t'\mathbf{V}_1, \tag{2.2-18}$$

where $r_{1,2}'=\left|\mathbf{r}_{1,2}'\right|$, $\hat{n}_{1,2}'$ is the dimensionless unit vector in the direction of $\mathbf{r}_{1,2}'$, $\mathbf{r}_{1,2}$ is given by (2.1-14) and is repeated below for convenience (also see Fig. 2.2-2),

$$\mathbf{r}_{1,2} = \mathbf{r}_2 - \mathbf{r}_1, \tag{2.1-14}$$

and $\Delta t'$ is given by (2.2-6). Therefore, (2.2-15) and (2.2-16) can be rewritten as

$$y_M(t,\mathbf{r}_2) = S_0' H_M\left(t,\mathbf{r}_{1,2}' \middle| f\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.2-19}$$

and

$$H_M\left(t,\mathbf{r}_{1,2}' \middle| f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_{1,2}'\right|\right)}{4\pi\left|\mathbf{r}_{1,2}'\right|}. \tag{2.2-20}$$

Let us now begin the process of obtaining a final expression for the time-harmonic velocity potential incident upon the receiver. Substituting (2.2-14) into (2.2-19) yields

$$y_M(t,\mathbf{r}_2) = S_0 H_M\left(t',\mathbf{r}_{0,1}' \middle| f\right) g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right) H_M\left(t,\mathbf{r}_{1,2}' \middle| f\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.2-21}$$

or, equivalently,

$$y_M(t,\mathbf{r}_2) = S_0 H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.2-22}$$

where

$$\begin{aligned} H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right) &= H_M\left(t',\mathbf{r}_{0,1}' \middle| f\right) g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right) H_M\left(t,\mathbf{r}_{1,2}' \middle| f\right) \\ &= g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}_{0,1}'\right| + \left|\mathbf{r}_{1,2}'\right|\right)\right]}{16\pi^2\left|\mathbf{r}_{0,1}'\right|\left|\mathbf{r}_{1,2}'\right|} \\ &= g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}_1 - \mathbf{r}_0 + \Delta t'\mathbf{V}_1\right| + \left|\mathbf{r}_2 - \mathbf{r}_1 - \Delta t'\mathbf{V}_1\right|\right)\right]}{16\pi^2\left|\mathbf{r}_1 - \mathbf{r}_0 + \Delta t'\mathbf{V}_1\right|\left|\mathbf{r}_2 - \mathbf{r}_1 - \Delta t'\mathbf{V}_1\right|}, \end{aligned} \tag{2.2-23}$$

$\Delta t'$ is given by (2.2-6), and

$$t = t' + \frac{\left|\mathbf{r}_{1,2}'\right|}{c}, \tag{2.2-24}$$

or

$$\boxed{t' = t - \frac{\left|\mathbf{r}_{1,2}'\right|}{c},} \tag{2.2-25}$$

where $\mathbf{r}'_{1,2}$ is given by (2.2-18). Note that if the bistatic scattering problem shown in Fig. 2.2-1 corresponded to transmission through a space-invariant filter, then the complex frequency response given by (2.2-23) would be a function of the vector spatial difference $\mathbf{r}_2 - \mathbf{r}_0$, which it is *not*.

In order to evaluate the complex frequency response given by (2.2-23), we must derive solutions for the unit vectors $\hat{n}'_{0,1}$ and $\hat{n}'_{1,2}$, and the ranges $|\mathbf{r}'_{0,1}|$ and $|\mathbf{r}'_{1,2}|$ in terms of known quantities. Let us begin with $|\mathbf{r}'_{0,1}|$. Since

$$|\mathbf{r}'_{0,1}|^2 = \mathbf{r}'_{0,1} \bullet \mathbf{r}'_{0,1}, \tag{2.2-26}$$

substituting (2.2-10) into (2.2-26) yields

$$|\mathbf{r}'_{0,1}|^2 = (\mathbf{r}_{0,1} + \Delta t' \mathbf{V}_1) \bullet (\mathbf{r}_{0,1} + \Delta t' \mathbf{V}_1). \tag{2.2-27}$$

Expanding the right-hand side of (2.2-27) and taking the square root of both sides of the resulting equation yields

$$\boxed{|\mathbf{r}'_{0,1}| = r_{0,1}\left[1 + \frac{2}{r_{0,1}}(\hat{n}_{0,1} \bullet \mathbf{V}_1)\Delta t' + \left(\frac{V_1 \Delta t'}{r_{0,1}}\right)^2\right]^{1/2},} \tag{2.2-28}$$

or

$$\boxed{|\mathbf{r}'_{0,1}| = r_{0,1}\left\{1 + 2\frac{V_1 \Delta t'}{r_{0,1}}\left[(\hat{n}_{0,1} \bullet \hat{n}_{V_1}) + \frac{1}{2}\frac{V_1 \Delta t'}{r_{0,1}}\right]\right\}^{1/2}.} \tag{2.2-29}$$

Although (2.2-28) and (2.2-29) are *exact* expressions for the range $|\mathbf{r}'_{0,1}|$, we cannot compute a value for $|\mathbf{r}'_{0,1}|$ until we derive an expression for $\Delta t'$ in terms of $|\mathbf{r}'_{0,1}|$. By referring to Fig. 2.2-2, it can be seen that

$$\boxed{\Delta t' = \frac{|\mathbf{r}'_{0,1}|}{c}, \qquad t' > t_m.} \tag{2.2-30}$$

Substituting (2.2-30) into (2.2-28) and squaring both sides of the resulting equation yields the following second-order polynomial

$$A|\mathbf{r}'_{0,1}|^2 - B|\mathbf{r}'_{0,1}| - C = 0, \tag{2.2-31}$$

with *exact* solution

$$\boxed{\left|\mathbf{r}'_{0,1}\right| = \frac{B \pm \sqrt{B^2 + 4AC}}{2A},} \tag{2.2-32}$$

where

$$A = 1 - \left(\frac{V_1}{c}\right)^2, \tag{2.2-33}$$

$$B = 2\, r_{0,1} \frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c}, \tag{2.2-34}$$

and

$$C = r_{0,1}^2. \tag{2.2-35}$$

The solution given by (2.2-32) is the *constant* value of range between the point source and the discrete point scatterer when the transmitted acoustic field is *first* incident upon the discrete point scatterer at time instant t' after motion begins at time instant t_m where $t' > t_m$. The decision to use either the plus or minus sign in (2.2-32) is dictated by the fact that range must be positive. Let us solve for $\left|\mathbf{r}'_{1,2}\right|$ next.

Since

$$\left|\mathbf{r}'_{1,2}\right|^2 = \mathbf{r}'_{1,2} \bullet \mathbf{r}'_{1,2}, \tag{2.2-36}$$

substituting (2.2-18) into (2.2-36) yields

$$\left|\mathbf{r}'_{1,2}\right|^2 = (\mathbf{r}_{1,2} - \Delta t' \mathbf{V}_1) \bullet (\mathbf{r}_{1,2} - \Delta t' \mathbf{V}_1). \tag{2.2-37}$$

Expanding the right-hand side of (2.2-37) and taking the square root of both sides of the resulting equation yields

$$\boxed{\left|\mathbf{r}'_{1,2}\right| = r_{1,2}\left[1 - \frac{2}{r_{1,2}}(\hat{n}_{1,2} \bullet \mathbf{V}_1)\Delta t' + \left(\frac{V_1 \Delta t'}{r_{1,2}}\right)^2\right]^{1/2},} \tag{2.2-38}$$

or

$$\boxed{\left|\mathbf{r}'_{1,2}\right| = r_{1,2}\left\{1 - 2\frac{V_1 \Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_1}) - \frac{1}{2}\frac{V_1 \Delta t'}{r_{1,2}}\right]\right\}^{1/2}.} \tag{2.2-39}$$

Although (2.2-38) and (2.2-39) are *exact* expressions for the range $\left|\mathbf{r}'_{1,2}\right|$, we cannot compute a value for $\left|\mathbf{r}'_{1,2}\right|$ until we derive an appropriate expression for $\Delta t'$. We will first solve for a constant value for $\left|\mathbf{r}'_{1,2}\right|$ by expressing $\Delta t'$ in terms of the *known* constant range $\left|\mathbf{r}'_{0,1}\right|$ [see (2.2-32)]. We will then solve for a time-varying $\left|\mathbf{r}'_{1,2}\right|$ by expressing $\Delta t'$ exclusively in terms of $\left|\mathbf{r}'_{1,2}\right|$.

Since (2.2-30) already expresses $\Delta t'$ in terms of $\left|\mathbf{r}'_{0,1}\right|$, substituting (2.2-30) into (2.2-38) yields

$$\boxed{\left|\mathbf{r}'_{1,2}\right| = r_{1,2}\left[1 - 2\frac{(\hat{n}_{1,2} \bullet \mathbf{V}_1)}{c}\frac{\left|\mathbf{r}'_{0,1}\right|}{r_{1,2}} + \left(\frac{V_1}{c}\frac{\left|\mathbf{r}'_{0,1}\right|}{r_{1,2}}\right)^2\right]^{1/2},} \tag{2.2-40}$$

where $\left|\mathbf{r}'_{0,1}\right|$ is given by (2.2-32). The solution given by (2.2-40) is the *constant* value of range between the discrete point scatterer and the receiver when the scattered acoustic field is *first* incident upon the receiver at time instant t after motion begins at time instant t_m where $t > t' > t_m$.

As we previously mentioned, we will now solve for a time-varying $\left|\mathbf{r}'_{1,2}\right|$ by expressing $\Delta t'$ exclusively in terms of $\left|\mathbf{r}'_{1,2}\right|$. We begin by substituting (2.2-25) into (2.2-6) which yields

$$\Delta t' = t - t_m - \frac{\left|\mathbf{r}'_{1,2}\right|}{c}, \qquad t > t' > t_m, \tag{2.2-41}$$

and upon substituting (2.2-4) into (2.2-41), we obtain

$$\boxed{\Delta t' = \Delta t - \frac{\left|\mathbf{r}'_{1,2}\right|}{c}, \qquad t > t' > t_m.} \tag{2.2-42}$$

The solution for $\left|\mathbf{r}'_{1,2}\right|$ can be obtained by substituting (2.2-42) into (2.2-38) and squaring both sides of the resulting equation. Doing so yields the following second-order polynomial

$$\mathcal{A}\left|\mathbf{r}'_{1,2}(t)\right|^2 + \mathcal{B}(t)\left|\mathbf{r}'_{1,2}(t)\right| - \mathcal{C}(t) = 0, \qquad t \geq t_m + \tau, \tag{2.2-43}$$

with *exact* solution

$$\boxed{\left|\mathbf{r}'_{1,2}(t)\right| = \frac{-\mathcal{B}(t) \pm \sqrt{\mathcal{B}^2(t) + 4\mathcal{A}\mathcal{C}(t)}}{2\mathcal{A}}, \qquad t \geq t_m + \tau,} \tag{2.2-44}$$

where

$$\mathcal{A} = 1 - \left(\frac{V_1}{c}\right)^2, \tag{2.2-45}$$

$$\mathcal{B}(t) = 2\frac{V_1^2}{c}\Delta t - 2r_{1,2}\frac{\hat{n}_{1,2} \bullet \mathbf{V}_1}{c}, \tag{2.2-46}$$

$$\mathcal{C}(t) = V_1^2(\Delta t)^2 - 2r_{1,2}(\hat{n}_{1,2} \bullet \mathbf{V}_1)\Delta t + r_{1,2}^2, \tag{2.2-47}$$

$$\Delta t = t - t_m, \qquad t \geq t_m + \tau, \tag{2.2-48}$$

and (see Fig. 2.2-2)

$$\boxed{\tau = \frac{\left|\mathbf{r}'_{0,1}\right|}{c} + \frac{\left|\mathbf{r}'_{1,2}\right|}{c}} \tag{2.2-49}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver after motion begins at time instant t_m) where the *constant* values of range $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}'_{1,2}\right|$ are given by (2.2-32) and (2.2-40), respectively. It is important to note that if (2.2-43) is evaluated at $t = t_m + \tau$, then it can be shown that

$$\left|\mathbf{r}'_{1,2}(t_m + \tau)\right| = \left|\mathbf{r}'_{1,2}\right|, \tag{2.2-50}$$

where $\left|\mathbf{r}'_{1,2}\right|$ is given by (2.2-40).

Now that we have an exact solution for $\left|\mathbf{r}'_{1,2}(t)\right|$ as given by (2.2-44), we can use it to obtain an *exact* solution for the time-varying range $\left|\mathbf{r}'_{0,1}(t)\right|$ as follows. With the use of (2.2-30) and (2.2-42), we can write that

$$\boxed{\left|\mathbf{r}'_{0,1}(t)\right| = c\Delta t - \left|\mathbf{r}'_{1,2}(t)\right|, \qquad t \geq t_m + \tau,} \tag{2.2-51}$$

where Δt is given by (2.2-48), $\left|\mathbf{r}'_{1,2}(t)\right|$ is given by (2.2-44), and τ is given by (2.2-49). Equation (2.2-51) indicates that if we are given a value of Δt, which determines the value of $\left|\mathbf{r}'_{1,2}(t)\right|$, then we can use those two values to compute what $\left|\mathbf{r}'_{0,1}(t)\right|$ must have been - we are working backwards from a value of Δt to a value for $\left|\mathbf{r}'_{1,2}(t)\right|$ to a value for $\left|\mathbf{r}'_{0,1}(t)\right|$. Note that if we evaluate (2.2-51) at $t = t_m + \tau$, then

$$\left|\mathbf{r}'_{0,1}(t_m + \tau)\right| = c\tau - \left|\mathbf{r}'_{1,2}(t_m + \tau)\right|, \tag{2.2-52}$$

and upon substituting (2.2-49) and (2.2-50) into (2.2-52), we obtain

$$\left|\mathbf{r}'_{0,1}(t_m+\tau)\right| = \left|\mathbf{r}'_{0,1}\right|, \tag{2.2-53}$$

where $\left|\mathbf{r}'_{0,1}\right|$ is given by (2.2-32).

Let us next solve for the unit vectors $\hat{n}'_{0,1}$ and $\hat{n}'_{1,2}$ so that we can evaluate the scattering function $g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)$ and, hence, the complex frequency response of the ocean. Let us begin with the unit vector $\hat{n}'_{0,1}$. From (2.2-10),

$$\hat{n}'_{0,1} = \frac{1}{\left|\mathbf{r}'_{0,1}\right|}\left(\mathbf{r}_{0,1} + \Delta t'\,\mathbf{V}_1\right). \tag{2.2-54}$$

Substituting (2.2-30) into (2.2-54) yields

$$\boxed{\hat{n}'_{0,1}(t) = \frac{1}{\left|\mathbf{r}'_{0,1}(t)\right|}\left[\mathbf{r}_{0,1} + \frac{\left|\mathbf{r}'_{0,1}(t)\right|}{c}\mathbf{V}_1\right], \qquad t \ge t_m + \tau,} \tag{2.2-55}$$

where $\left|\mathbf{r}'_{0,1}(t)\right|$ is given by (2.2-51), $\mathbf{r}_{0,1}$ is given by (2.1-6), and τ is given by (2.2-49). Note that if we evaluate (2.2-55) at $t = t_m + \tau$, then

$$\hat{n}'_{0,1}(t_m+\tau) = \frac{1}{\left|\mathbf{r}'_{0,1}(t_m+\tau)\right|}\left[\mathbf{r}_{0,1} + \frac{\left|\mathbf{r}'_{0,1}(t_m+\tau)\right|}{c}\mathbf{V}_1\right], \tag{2.2-56}$$

and upon substituting (2.2-53) and (2.2-30) into (2.2-56), we obtain

$$\hat{n}'_{0,1}(t_m+\tau) = \hat{n}'_{0,1}, \tag{2.2-57}$$

where $\hat{n}'_{0,1}$ is given by (2.2-54). And by referring to (2.1-35) and (2.1-36), we can write that

$$\boxed{\theta'_{0,1}(t) = \cos^{-1} w'_{0,1}(t), \qquad t \ge t_m + \tau,} \tag{2.2-58}$$

and

$$\boxed{\psi'_{0,1}(t) = \tan^{-1}\left(\frac{v'_{0,1}(t)}{u'_{0,1}(t)}\right), \qquad t \ge t_m + \tau,} \tag{2.2-59}$$

where $u'_{0,1}(t)$, $v'_{0,1}(t)$, and $w'_{0,1}(t)$ are the dimensionless, time-varying direction cosines with respect to the X, Y, and Z axes, respectively, associated with the time-varying unit vector $\hat{n}'_{0,1}(t)$ given by (2.2-55), and τ is given by (2.2-49). Equations (2.2-58) and (2.2-59) are the *angles of incidence* at the discrete point scatterer.

From (2.2-18),

$$\hat{n}'_{1,2} = \frac{1}{\left|\mathbf{r}'_{1,2}\right|}\left(\mathbf{r}_{1,2} - \Delta t' \, \mathbf{V}_1\right). \tag{2.2-60}$$

Substituting (2.2-42) into (2.2-60) yields

$$\boxed{\hat{n}'_{1,2}(t) = \frac{1}{\left|\mathbf{r}'_{1,2}(t)\right|}\left\{\mathbf{r}_{1,2} - \left[\Delta t - \frac{\left|\mathbf{r}'_{1,2}(t)\right|}{c}\right]\mathbf{V}_1\right\}, \qquad t \geq t_m + \tau,} \tag{2.2-61}$$

where $\left|\mathbf{r}'_{1,2}(t)\right|$ is given by (2.2-44), $\mathbf{r}_{1,2}$ is given by (2.1-14), Δt is given by (2.2-48), and τ is given by (2.2-49). Note that if we evaluate (2.2-61) at $t = t_m + \tau$, then

$$\hat{n}'_{1,2}(t_m + \tau) = \frac{1}{\left|\mathbf{r}'_{1,2}(t_m + \tau)\right|}\left\{\mathbf{r}_{1,2} - \left[\tau - \frac{\left|\mathbf{r}'_{1,2}(t_m + \tau)\right|}{c}\right]\mathbf{V}_1\right\}, \tag{2.2-62}$$

and upon substituting (2.2-49), (2.2-50), and (2.2-30) into (2.2-62), we obtain

$$\hat{n}'_{1,2}(t_m + \tau) = \hat{n}'_{1,2}, \tag{2.2-63}$$

where $\hat{n}'_{1,2}$ is given by (2.2-60). And by referring to (2.1-44) and (2.1-45), we can write that

$$\boxed{\theta'_{1,2}(t) = \cos^{-1} w'_{1,2}(t), \qquad t \geq t_m + \tau,} \tag{2.2-64}$$

and

$$\boxed{\psi'_{1,2}(t) = \tan^{-1}\left(\frac{v'_{1,2}(t)}{u'_{1,2}(t)}\right), \qquad t \geq t_m + \tau,} \tag{2.2-65}$$

where $u'_{1,2}(t)$, $v'_{1,2}(t)$, and $w'_{1,2}(t)$ are the dimensionless, time-varying direction cosines with respect to the X, Y, and Z axes, respectively, associated with the time-varying unit vector $\hat{n}'_{1,2}(t)$ given by (2.2-61), and τ is given by (2.2-49). Equations (2.2-64) and (2.2-65) are the *angles of scatter* at the receiver.

Note that since the unit vectors $\hat{n}'_{0,1}(t)$ and $\hat{n}'_{1,2}(t)$ given by (2.2-55) and (2.2-61), respectively, are functions of time, the scattering function is also a function of time, that is,

$$g_1\left(f, \hat{n}'_{0,1}, \hat{n}'_{1,2}\right) = g_1\left(f, \hat{n}'_{0,1}(t), \hat{n}'_{1,2}(t)\right), \qquad t \geq t_m + \tau. \tag{2.2-66}$$

With the use of (2.2-22) and (2.2-23), and by replacing the real wavenumber k in (2.2-23) with the complex wavenumber K given by (2.1-20), we can *summarize* our results as follows: for the bistatic scattering problem shown in Fig. 2.2-1, the time-harmonic velocity potential in squared-

meters per second incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$, due to a time-harmonic point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a moving discrete point scatterer initially at $\mathbf{r}_1 = (x_1, y_1, z_1)$, is given by

$$\boxed{y_M(t,\mathbf{r}_2) = S_0 H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right)\exp(+j2\pi f t), \qquad t \geq t_m + \tau,} \tag{2.2-67}$$

where S_0 is the source strength in cubic meters per second,

$$\boxed{H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right) = g_1\left(f,\hat{n}'_{0,1}(t),\hat{n}'_{1,2}(t)\right)\frac{\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}(t)\right| + \left|\mathbf{r}'_{1,2}(t)\right|\right)\right]}{16\pi^2\left|\mathbf{r}'_{0,1}(t)\right|\left|\mathbf{r}'_{1,2}(t)\right|}\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}(t)\right| + \left|\mathbf{r}'_{1,2}(t)\right|\right)\right]}$$

(2.2-68)

is the time-variant, space-variant, complex frequency response of the ocean at frequency f hertz,

$$\boxed{g_1\left(f,\hat{n}'_{0,1}(t),\hat{n}'_{1,2}(t)\right) \equiv g_1\left(f,\theta'_{0,1}(t),\psi'_{0,1}(t),\theta'_{1,2}(t),\psi'_{1,2}(t)\right)} \tag{2.2-69}$$

is the scattering function of the discrete point scatterer in meters, the angles of incidence $\theta'_{0,1}(t)$ and $\psi'_{0,1}(t)$ are given by (2.2-58) and (2.2-59), respectively, the angles of scatter $\theta'_{1,2}(t)$ and $\psi'_{1,2}(t)$ are given by (2.2-64) and (2.2-65), respectively, $\alpha(f)$ is the real, frequency-dependent, attenuation coefficient in nepers per meter, $\left|\mathbf{r}'_{0,1}(t)\right|$ is given by (2.2-51), $\left|\mathbf{r}'_{1,2}(t)\right|$ is given by (2.2-44), the real wavenumber k in radians per meter is given by (2.1-5), and the time delay τ in seconds is given by (2.2-49).

Let us next relate the scattering function, the differential scattering cross section, and target strength of the discrete point scatterer. The *target strength* (TS) is defined as follows [6]:

$$\boxed{\mathrm{TS} \triangleq 10\log_{10}\left[\frac{\sigma_d\left(f,\hat{n}'_{0,1}(t),\hat{n}'_{1,2}(t)\right)}{A_{\mathrm{ref}}}\right] \mathrm{dB\ re}\ A_{\mathrm{ref}}, \qquad t \geq t_m + \tau,} \tag{2.2-70}$$

where [5, 6]

$$\boxed{\sigma_d\left(f,\hat{n}'_{0,1}(t),\hat{n}'_{1,2}(t)\right) \triangleq \lim_{r'_{1,2}(t)\to\infty}\left[\frac{[r'_{1,2}(t)]^2 I_{avg_s}(\mathbf{r}_2)}{I_{avg_i}\left(\mathbf{r}'_1(t)\right)}\right] = \frac{\left|g_1\left(f,\hat{n}'_{0,1}(t),\hat{n}'_{1,2}(t)\right)\right|^2}{(4\pi)^2}, \qquad t \geq t_m + \tau,}$$

(2.2-71)

is the *differential scattering cross section* with units of squared meters, $I_{avg_i}\left(\mathbf{r}'_1(t)\right)$ and $I_{avg_s}(\mathbf{r}_2)$ are the *time-average*, *incident* and *scattered intensities*, respectively, with units of watts per squared meter, $g_1\left(f,\hat{n}'_{0,1}(t),\hat{n}'_{1,2}(t)\right)$ is the scattering function of the discrete point scatterer with units of

meters, A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to 1 m^2, and the time delay τ in seconds is given by (2.2-49).

Example 2.2-1 Monostatic Scattering Geometry

In this example the discrete point scatterer is in motion with constant velocity vector $\mathbf{V}_1$, but the scattering geometry is monostatic versus bistatic. For a *monostatic* (*backscatter*) scattering geometry, both the transmitter and receiver are located at the same position, that is,

$$\mathbf{r}_2 = \mathbf{r}_0 . \tag{2.2-72}$$

Substituting (2.2-72) into (2.1-14) yields

$$\mathbf{r}_{1,2} = -\mathbf{r}_{0,1} , \tag{2.2-73}$$

where $\mathbf{r}_{0,1}$ is given by (2.1-6). Therefore,

$$r_{1,2} = r_{0,1} \tag{2.2-74}$$

and

$$\hat{n}_{1,2} = -\hat{n}_{0,1} . \tag{2.2-75}$$

And upon substituting (2.1-73) into (2.2-18), we obtain

$$\mathbf{r}'_{1,2} = -\mathbf{r}'_{0,1} , \tag{2.2-76}$$

where $\mathbf{r}'_{0,1}$ is given by (2.2-10). Therefore,

$$\left|\mathbf{r}'_{1,2}\right| = \left|\mathbf{r}'_{0,1}\right| \tag{2.2-77}$$

and

$$\hat{n}'_{1,2} = -\hat{n}'_{0,1} , \tag{2.2-78}$$

and upon generalizing,

$$\left|\mathbf{r}'_{1,2}(t)\right| = \left|\mathbf{r}'_{0,1}(t)\right| \tag{2.2-79}$$

and

$$\hat{n}'_{1,2}(t) = -\hat{n}'_{0,1}(t) . \tag{2.2-80}$$

With the use of (2.2-79) and (2.2-80), the time-variant, space-variant, complex frequency response of the ocean given by (2.2-68) reduces to

$$H_M\left(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0\right)=g_1\left(f,\hat{n}'_{0,1}(t),-\hat{n}'_{0,1}(t)\right)\frac{\exp\left[-2\alpha(f)\left|\mathbf{r}'_{0,1}(t)\right|\right]}{\left(4\pi\left|\mathbf{r}'_{0,1}(t)\right|\right)^2}\exp\left[-j2k\left|\mathbf{r}'_{0,1}(t)\right|\right],\qquad t\geq t_m+\tau,$$

(2.2-81)

where

$$g_1\left(f,\hat{n}'_{0,1}(t),-\hat{n}'_{0,1}(t)\right)\equiv g_1\left(f,\theta'_{0,1}(t),\psi'_{0,1}(t),\pi-\theta'_{0,1}(t),\pi+\psi'_{0,1}(t)\right), \tag{2.2-82}$$

$\theta'_{0,1}(t)$ and $\psi'_{0,1}(t)$ are the angles of incidence given by (2.2-58) and (2.2-59), respectively, and $\theta'_{1,2}(t)=\pi-\theta'_{0,1}(t)$ and $\psi'_{1,2}(t)=\pi+\psi'_{0,1}(t)$ are the angles of scatter. Substituting (2.2-79) into (2.2-44), (2.2-74) and (2.2-75) into (2.2-46) and (2.2-47), and (2.2-77) into (2.2-49) yields

$$\left|\mathbf{r}'_{0,1}(t)\right|=\frac{-\mathcal{B}(t)\pm\sqrt{\mathcal{B}^2(t)+4\mathcal{A}\mathcal{C}(t)}}{2\mathcal{A}},\qquad t\geq t_m+\tau, \tag{2.2-83}$$

where

$$\mathcal{A}=1-\left(\frac{V_1}{c}\right)^2, \tag{2.2-84}$$

$$\mathcal{B}(t)=2\frac{V_1^2}{c}\Delta t+2\,r_{0,1}\frac{\hat{n}_{0,1}\bullet\mathbf{V}_1}{c}, \tag{2.2-85}$$

$$\mathcal{C}(t)=V_1^2(\Delta t)^2+2\,r_{0,1}(\hat{n}_{0,1}\bullet\mathbf{V}_1)\Delta t+r_{0,1}^2, \tag{2.2-86}$$

$$\Delta t=t-t_m,\qquad t\geq t_m+\tau, \tag{2.2-87}$$

and

$$\tau=\frac{2\left|\mathbf{r}'_{0,1}\right|}{c} \tag{2.2-88}$$

is the time delay in seconds, where the *constant* value of range $\left|\mathbf{r}'_{0,1}\right|$ is given by (2.2-32).

2.3 All Three Platforms In Motion

In this section we will analyze the bistatic scattering problem shown in Fig. 2.3-1. All three platforms - the transmitter, discrete point scatterer, and receiver - are in motion. Motion corresponds to a *time-variant* problem. As mentioned in the Introduction, the three platforms will be treated as being in an unbounded, homogeneous ocean medium. Although the propagation of sound between the source and discrete point scatterer, and between the discrete point scatterer and receiver can be treated as transmission through linear, *time-variant, space-invariant* filters; the overall solution for this bistatic scattering problem corresponds to transmission through a linear, *time-variant, space-variant* filter. The presence of a discrete point scatterer in an unbounded, homogeneous fluid

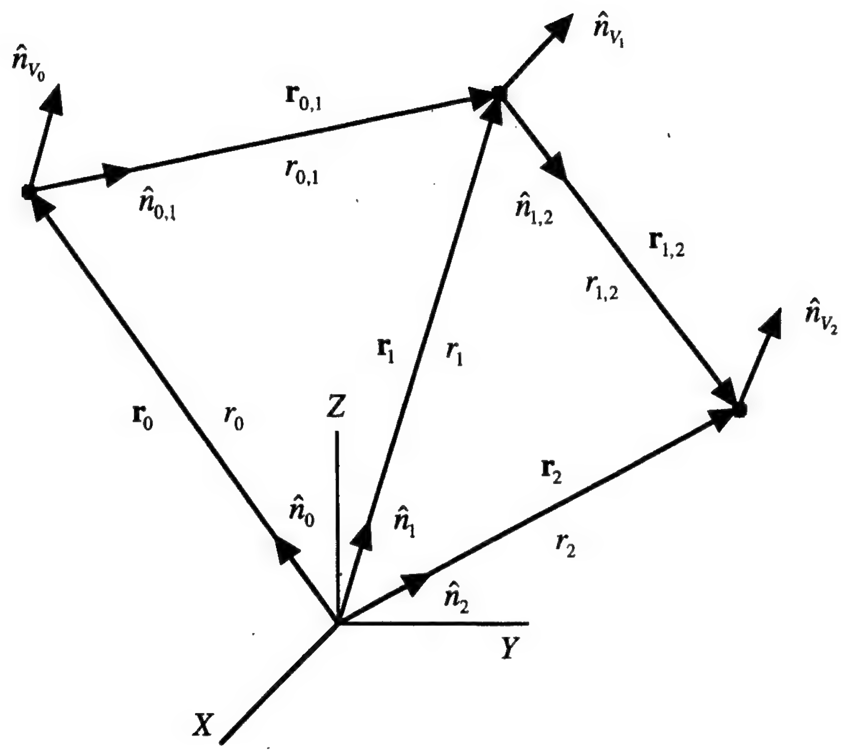

Figure 2.3-1. Bistatic scattering geometry when motion begins at time $t = t_m$ seconds. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. All three platforms are in motion.

medium (i.e., a fluid medium with constant speed of sound and ambient density) causes the medium to be space-variant.

The *velocity vectors* of the transmitter, $\mathbf{V}_0$, the discrete point scatterer, $\mathbf{V}_1$, and the receiver, $\mathbf{V}_2$, are given by

$$\mathbf{V}_0 = V_0 \hat{n}_{V_0}, \tag{2.3-1}$$

$$\mathbf{V}_1 = V_1 \hat{n}_{V_1}, \tag{2.3-2}$$

and

$$\mathbf{V}_2 = V_2 \hat{n}_{V_2}, \tag{2.3-3}$$

where V_0, V_1, and V_2 are the *speeds* in meters per second of the transmitter, the discrete point scatterer, and the receiver, respectively, and $\hat{n}_{V_0}$, $\hat{n}_{V_1}$, and $\hat{n}_{V_2}$ are the dimensionless unit vectors in the directions of $\mathbf{V}_0$, $\mathbf{V}_1$, and $\mathbf{V}_2$, respectively. The velocity vectors given by (2.3-1) through (2.3-3) are *constant*, that is, the speeds and directions are *constants* - there is *no* acceleration. Motion begins at time $t = t_m$ seconds. We will model the propagation of sound from the time motion begins.

Since all three platforms are now in motion, the position vectors from the origin to the transmitter, discrete point scatterer, and receiver - denoted by $\mathcal{R}_0(t)$, $\mathcal{R}_1(t)$, and $\mathcal{R}_2(t)$, respectively - are functions of time given by

$$\mathcal{R}_0(t) = \mathbf{r}_0 + \Delta t\, \mathbf{V}_0, \qquad t \geq t_m, \tag{2.3-4}$$

$$\mathcal{R}_1(t) = \mathbf{r}_1 + \Delta t\, \mathbf{V}_1, \qquad t \geq t_m, \tag{2.3-5}$$

and

$$\mathcal{R}_2(t) = \mathbf{r}_2 + \Delta t\, \mathbf{V}_2, \qquad t \geq t_m, \tag{2.3-6}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$, $\mathbf{r}_1 = (x_1, y_1, z_1)$, and $\mathbf{r}_2 = (x_2, y_2, z_2)$ are the position vectors from the origin to the transmitter, discrete point scatterer, and receiver, respectively, when motion begins (see Fig. 2.3-1), and

$$\Delta t = t - t_m, \qquad t \geq t_m. \tag{2.3-7}$$

Note that $\mathcal{R}_0(t_m) = \mathbf{r}_0$, $\mathcal{R}_1(t_m) = \mathbf{r}_1$, and $\mathcal{R}_2(t_m) = \mathbf{r}_2$. Instead of trying to solve this bistatic scattering problem directly with all three platforms in motion, we will first create an *equivalent problem* involving the transmitter and the discrete point scatterer where we can treat the transmitter (sound source) as being *motionless*. We will then create a *second equivalent problem* involving the discrete point scatterer and the receiver where we can treat the discrete point scatterer (acting as a sound source) as being *motionless*. This can be accomplished by working with *relative velocity vectors*.

When motion begins at time t_m, the *scalar* component of $\mathbf{V}_0$ in the direction of $\mathbf{V}_1$ is given by (see Fig. 2.3-2)

$$\hat{n}_{V_1} \bullet \mathbf{V}_0 = \hat{n}_{V_1} \bullet V_0 \hat{n}_{V_0} = V_0 (\hat{n}_{V_1} \bullet \hat{n}_{V_0}). \tag{2.3-8}$$

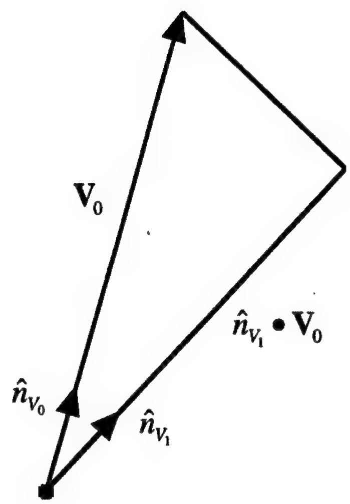

Figure 2.3-2. Scalar component of $\mathbf{V}_0$ in the direction of $\mathbf{V}_1$.

Therefore, the velocity vector of the discrete point scatterer *relative* to the velocity vector of the transmitter *in the direction of the velocity vector of the discrete point scatterer* $\hat{n}_{V_1}$ is given by

$$\boxed{\mathbf{V}_{1,0} = \mathbf{V}_1 - (\hat{n}_{V_1} \bullet \mathbf{V}_0)\hat{n}_{V_1} = \left[V_1 - V_0(\hat{n}_{V_1} \bullet \hat{n}_{V_0})\right]\hat{n}_{V_1}.} \tag{2.3-9}$$

By using the *relative velocity vector* $\mathbf{V}_{1,0}$, the transmitter (sound source) can be treated as being *motionless*. Therefore, when motion begins, the source distribution $x_M(t,\mathbf{r})$ at time t and position $\mathbf{r} = (x, y, z)$ will be treated as a *motionless*, *time-harmonic*, *point source* with units of inverse seconds, that is, let

$$x_M(t,\mathbf{r}) = S_0\delta(\mathbf{r} - \mathbf{r}_0)\exp(+j2\pi ft), \tag{2.3-10}$$

where S_0 is the *source strength* in cubic meters per second, the impulse function $\delta(\mathbf{r} - \mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$, and f is frequency in hertz. The sound source has been turned on forever, that is, since $t = -\infty$. In addition, since we are now working with the relative velocity vector $\mathbf{V}_{1,0}$, we need to introduce the new position vector

$$\mathcal{R}_1^{(1,0)}(t) = \mathbf{r}_1 + \Delta t\,\mathbf{V}_{1,0}, \qquad t \geq t_m, \tag{2.3-11}$$

where Δt is given by (2.3-7). Compare (2.3-11) with (2.3-5). Note that $\mathcal{R}_1^{(1,0)}(t_m) = \mathbf{r}_1$. Also note that if the transmitter is *not* in motion, then $\mathbf{V}_0 = \mathbf{0}$, and as a result, $\mathbf{V}_{1,0} = \mathbf{V}_1$ [see (2.3-9)] and $\mathcal{R}_1^{(1,0)}(t) = \mathcal{R}_1(t)$ [see (2.3-11) and (2.3-5)]. And if the discrete point scatterer is *not* in motion, then $\mathbf{V}_1 = \mathbf{0}$ and the relative velocity vector $\mathbf{V}_{1,0}$ given by (2.3-9) is *undefined*. In this case we set $\mathbf{V}_{1,0} = \mathbf{0}$, and as a result, $\mathcal{R}_1^{(1,0)}(t) = \mathcal{R}_1(t) = \mathbf{r}_1$.

When the transmitted acoustic field is first incident upon the discrete point scatterer at some time t' seconds where $t' > t_m$, the position vector from the origin to the discrete point scatterer is given by [see (2.3-11) and Fig. 2.3-3]

$$\mathbf{r}_1' = \mathcal{R}_1^{(1,0)}(t') = \mathbf{r}_1 + \Delta t'\,\mathbf{V}_{1,0}, \qquad t' > t_m, \tag{2.3-12}$$

where

$$\Delta t' = t' - t_m, \qquad t' > t_m. \tag{2.3-13}$$

The propagation of sound between the source and discrete point scatterer can be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the discrete point scatterer at time t' and position $\mathbf{r}_1' = (x_1', y_1', z_1')$ is given by [7]

$$y_M(t',\mathbf{r}_1') = S_0 H_M\left(t', \mathbf{r}_1' - \mathbf{r}_0 \,|\, f\right)\exp(+j2\pi ft'), \qquad t' > t_m, \tag{2.3-14}$$

where

$$H_M\left(t', \mathbf{r}_1' - \mathbf{r}_0 \,|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_1' - \mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1' - \mathbf{r}_0\right|} \tag{2.3-15}$$

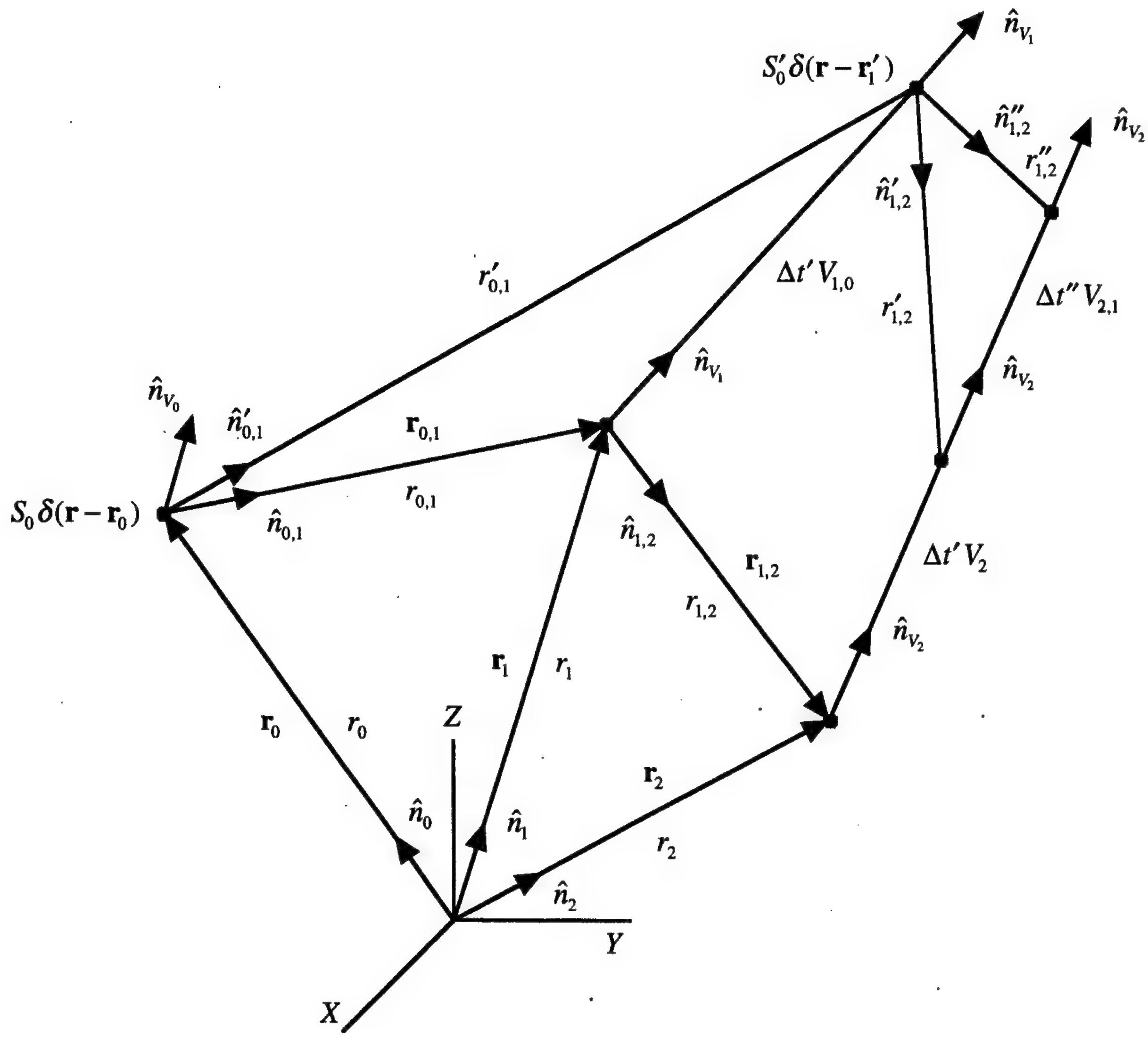

Figure 2.3-3. Bistatic scattering geometry when the transmitted acoustic field is first incident upon the discrete point scatterer at time t' seconds and when the scattered acoustic field is first incident upon the receiver at time t seconds where $t > t' > t_m$. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. All three platforms are in motion.

is the *time-variant, space-invariant, complex frequency response* of the ocean at frequency f hertz, and k is the wavenumber in radians per meter given by (2.1-5) and is repeated below for convenience:

$$k = 2\pi f / c = 2\pi / \lambda . \tag{2.1-5}$$

By referring to Fig. 2.3-3, we can express the position vector from the point source to the discrete point scatterer at time t' as

$$\mathbf{r}'_{0,1} = \mathbf{r}'_1 - \mathbf{r}_0, \tag{2.3-16}$$

and upon substituting (2.3-12) into (2.3-16), we obtain

$$\mathbf{r}'_{0,1} = r'_{0,1}\hat{n}'_{0,1} = \mathbf{r}_{0,1} + \Delta t' \mathbf{V}_{1,0}, \tag{2.3-17}$$

where $r'_{0,1} = \left|\mathbf{r}'_{0,1}\right|$, $\hat{n}'_{0,1}$ is the dimensionless unit vector in the direction of $\mathbf{r}'_{0,1}$, $\mathbf{r}_{0,1}$ is given by (2.1-6) and is repeated below for convenience (also see Fig. 2.3-3),

$$\mathbf{r}_{0,1} = \mathbf{r}_1 - \mathbf{r}_0, \tag{2.1-6}$$

and $\Delta t'$ is given by (2.3-13). Therefore, (2.3-14) and (2.3-15) can be rewritten as

$$y_M(t', \mathbf{r}'_1) = S_0 H_M\left(t', \mathbf{r}'_{0,1} \middle| f\right) \exp(+j2\pi f t'), \qquad t' > t_m, \tag{2.3-18}$$

and

$$H_M\left(t', \mathbf{r}'_{0,1} \middle| f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}'_{0,1}\right|\right)}{4\pi\left|\mathbf{r}'_{0,1}\right|}. \tag{2.3-19}$$

Let us now create a similar equivalent problem involving the discrete point scatterer and the receiver where we can treat the discrete point scatterer (acting as a sound source) as being motionless. When the transmitted acoustic field is first incident upon the discrete point scatterer at time $t' > t_m$ seconds, the position vector from the origin to the receiver is given by [see (2.3-6) and Fig. 2.3-3]

$$\mathbf{r}'_2 = \mathcal{R}_2(t') = \mathbf{r}_2 + \Delta t' \mathbf{V}_2, \qquad t' > t_m, \tag{2.3-20}$$

where $\Delta t'$ is given by (2.3-13). Equation (2.3-20) indicates that after $\Delta t'$ seconds, the receiver - independent of the transmitter and the discrete point scatterer - travels an additional distance of $\Delta t' V_2$ meters in the direction $\hat{n}_{V_2}$ (see Fig. 2.3-3). Also at time t', the *scalar* component of $\mathbf{V}_1$ in the direction of $\mathbf{V}_2$ is given by (see Fig. 2.3-4)

$$\hat{n}_{V_2} \bullet \mathbf{V}_1 = \hat{n}_{V_2} \bullet V_1 \hat{n}_{V_1} = V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1}). \tag{2.3-21}$$

Therefore, the velocity vector of the receiver *relative* to the velocity vector of the discrete point scatterer *in the direction of the velocity vector of the receiver* $\hat{n}_{V_2}$ is given by

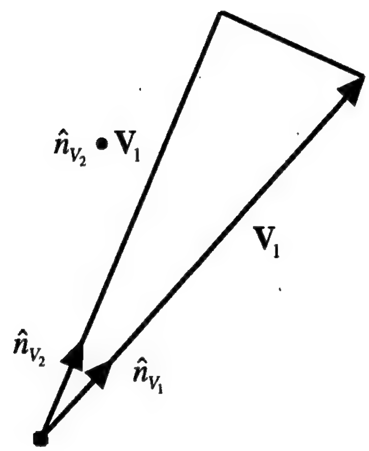

Figure 2.3-4. Scalar component of $\mathbf{V}_1$ in the direction of $\mathbf{V}_2$.

$$\boxed{\mathbf{V}_{2,1} = \mathbf{V}_2 - (\hat{n}_{V_2} \bullet \mathbf{V}_1)\hat{n}_{V_2} = \left[V_2 - V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_2}.} \tag{2.3-22}$$

By using the *relative velocity vector* $\mathbf{V}_{2,1}$, the discrete point scatterer (acting as a sound source) can be treated as being *motionless*. Therefore, in order to compute the acoustic signal incident upon the receiver, we treat the discrete point scatterer at time $t \geq t' > t_m$ and position $\mathbf{r}_1'$ as another *motionless, time-harmonic, point source* with units of inverse seconds, that is, let [see (2.3-10) and Fig. 2.3-3]

$$x_M'(t,\mathbf{r}) = S_0'\delta(\mathbf{r} - \mathbf{r}_1')\exp(+j2\pi ft), \tag{2.3-23}$$

where S_0' is the *source strength* in cubic meters per second, and the impulse function $\delta(\mathbf{r} - \mathbf{r}_1')$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_1' = (x_1', y_1', z_1')$, where $\mathbf{r}_1'$ is given by (2.3-12). The source strength S_0' will be given later.

Since we are now working with the relative velocity vector $\mathbf{V}_{2,1}$, we need to introduce the new position vector

$$\boldsymbol{\mathcal{R}}_2^{(2,1)}(t) = \mathbf{r}_2' + \Delta t'' \mathbf{V}_{2,1}, \qquad t \geq t' > t_m, \tag{2.3-24}$$

where

$$\Delta t'' = t - t', \qquad t \geq t' > t_m, \tag{2.3-25}$$

and $\mathbf{r}_2'$ is given by (2.3-20). Compare (2.3-24) with (2.3-6). Note that $\boldsymbol{\mathcal{R}}_2^{(2,1)}(t') = \mathbf{r}_2'$. Also note that if the discrete point scatterer is *not* in motion, then $\mathbf{V}_1 = \mathbf{0}$, and as a result, $\mathbf{V}_{2,1} = \mathbf{V}_2$ [see (2.3-22)] and $\boldsymbol{\mathcal{R}}_2^{(2,1)}(t) = \boldsymbol{\mathcal{R}}_2(t)$ for $t \geq t' > t_m$ [see (2.3-24), (2.3-20), and (2.3-6)]. And if the receiver is *not* in motion, then $\mathbf{V}_2 = \mathbf{0}$ and the relative velocity vector $\mathbf{V}_{2,1}$ given by (2.3-22) is *undefined*. In this case we set $\mathbf{V}_{2,1} = \mathbf{0}$, and as a result, $\boldsymbol{\mathcal{R}}_2^{(2,1)}(t) = \boldsymbol{\mathcal{R}}_2(t) = \mathbf{r}_2$ for $t \geq t' > t_m$.

When the scattered acoustic field is first incident upon the receiver at some time t seconds where $t > t' > t_m$, the position vector from the origin to the receiver is given by [see (2.3-24) and Fig. 2.3-3]

$$\mathbf{r}_2'' = \boldsymbol{\mathcal{R}}_2^{(2,1)}(t) = \mathbf{r}_2' + \Delta t'' \mathbf{V}_{2,1}, \qquad t > t' > t_m, \tag{2.3-26}$$

where $\Delta t''$ is given by (2.3-25). Equation (2.3-26) indicates that after $\Delta t''$ seconds, the receiver travels an additional distance of $\Delta t'' V_{2,1}$ meters in the direction $\hat{n}_{V_2}$ (see Fig. 2.3-3). By referring to Fig. 2.3-3, it can be seen that

$$\mathbf{r}_{1,2}'' = \mathbf{r}_2'' - \mathbf{r}_1', \tag{2.3-27}$$

where $\mathbf{r}_{1,2}''$ is the position vector between the discrete point scatterer and the receiver at time $t > t' > t_m$. Substituting (2.3-12) and (2.3-26) into (2.3-27) yields

$$\mathbf{r}_{1,2}'' = r_{1,2}''\hat{n}_{1,2}'' = \mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t'' \mathbf{V}_{2,1}, \tag{2.3-28}$$

where $r''_{1,2} = \left|\mathbf{r}''_{1,2}\right|$, $\hat{n}''_{1,2}$ is the dimensionless unit vector in the direction of $\mathbf{r}''_{1,2}$, $\mathbf{r}_{1,2}$ is given by (2.1-14) and is repeated below for convenience (also see Fig. 2.3-3),

$$\mathbf{r}_{1,2} = \mathbf{r}_2 - \mathbf{r}_1, \tag{2.1-14}$$

$\Delta t'$ is given by (2.3-13), and $\Delta t''$ is given by (2.3-25). Equation (2.3-28) can also be obtained directly from Fig. 2.3-3 - without the use of (2.3-27) - simply by using vector addition. Doing so yields

$$\begin{aligned} \mathbf{r}''_{1,2} = r''_{1,2}\hat{n}''_{1,2} &= -\Delta t' \mathbf{V}_{1,0} + \mathbf{r}_{1,2} + \Delta t' \mathbf{V}_2 + \Delta t'' \mathbf{V}_{2,1} \\ &= \mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t'' \mathbf{V}_{2,1}. \end{aligned} \tag{2.3-29}$$

The propagation of sound between the discrete point scatterer and receiver can also be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the receiver at time t and position $\mathbf{r}''_2 = (x''_2, y''_2, z''_2)$ due to a point source at $\mathbf{r}'_1 = (x'_1, y'_1, z'_1)$ is given by [7]

$$y_M(t,\mathbf{r}''_2) = S'_0 H_M\left(t,\mathbf{r}''_2 - \mathbf{r}'_1 \middle| f\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.3-30}$$

where

$$H_M\left(t,\mathbf{r}''_2 - \mathbf{r}'_1 \middle| f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}''_2 - \mathbf{r}'_1\right|\right)}{4\pi\left|\mathbf{r}''_2 - \mathbf{r}'_1\right|}. \tag{2.3-31}$$

With the use of (2.3-27), (2.3-30) and (2.3-31) can be rewritten as

$$y_M(t,\mathbf{r}''_2) = S'_0 H_M\left(t,\mathbf{r}''_{1,2} \middle| f\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.3-32}$$

and

$$H_M\left(t,\mathbf{r}''_{1,2} \middle| f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}''_{1,2}\right|\right)}{4\pi\left|\mathbf{r}''_{1,2}\right|}. \tag{2.3-33}$$

The source strength S'_0 is given by [see (2.3-18)]

$$S'_0 = S_0 H_M\left(t',\mathbf{r}'_{0,1} \middle| f\right) g_1\left(f,\hat{n}'_{0,1},\hat{n}''_{1,2}\right), \tag{2.3-34}$$

where $g_1\left(f,\hat{n}'_{0,1},\hat{n}''_{1,2}\right)$ is the *scattering function* of the discrete point scatterer with units of meters as discussed in Section 2.1. Since the transmitted acoustic field is first incident upon the discrete point scatterer at time t', the direction of wave propagation from the source to the scatterer is given by the dimensionless unit vector $\hat{n}'_{0,1}$ (see Fig. 2.3-3). Similarly, since the discrete point scatterer is being treated as another point source at time t' and position $\mathbf{r}'_1$, and the scattered acoustic field is first incident upon the receiver at time $t > t' > t_m$, the direction of wave propagation from the scatterer to the receiver is given by the dimensionless unit vector $\hat{n}''_{1,2}$ (see Fig. 2.3-3). Later in this section, we

will show that the scattering function is also a function of time because the unit vectors are actually *time-varying*. We will then show how to express the scattering function as a function of frequency and *time-varying angles of incidence and scatter* instead of time-varying unit vectors. The use of unit vectors is meant as a shorthand notation.

Let us now begin the process of obtaining a final expression for the time-harmonic velocity potential incident upon the receiver. Substituting (2.3-34) into (2.3-32) yields

$$y_M(t,\mathbf{r}_2'') = S_0 H_M\left(t',\mathbf{r}_{0,1}'\middle|f\right) g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right) H_M\left(t,\mathbf{r}_{1,2}''\middle|f\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.3-35}$$

or, equivalently,

$$y_M(t,\mathbf{r}_2'') = S_0 H_M\left(t,\mathbf{r}_2''\middle|f,\mathbf{r}_0\right)\exp(+j2\pi f t), \qquad t > t' > t_m, \tag{2.3-36}$$

where

$$\begin{aligned} H_M\left(t,\mathbf{r}_2''\middle|f,\mathbf{r}_0\right) &= H_M\left(t',\mathbf{r}_{0,1}'\middle|f\right) g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right) H_M\left(t,\mathbf{r}_{1,2}''\middle|f\right) \\ &= g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}_{0,1}'\right|+\left|\mathbf{r}_{1,2}''\right|\right)\right]}{16\pi^2\left|\mathbf{r}_{0,1}'\right|\left|\mathbf{r}_{1,2}''\right|} \\ &= g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}_1-\mathbf{r}_0+\Delta t'\mathbf{V}_{1,0}\right|+\left|\mathbf{r}_2-\mathbf{r}_1+\Delta t'(\mathbf{V}_2-\mathbf{V}_{1,0})+\Delta t''\mathbf{V}_{2,1}\right|\right)\right]}{16\pi^2\left|\mathbf{r}_1-\mathbf{r}_0+\Delta t'\mathbf{V}_{1,0}\right|\left|\mathbf{r}_2-\mathbf{r}_1+\Delta t'(\mathbf{V}_2-\mathbf{V}_{1,0})+\Delta t''\mathbf{V}_{2,1}\right|}, \end{aligned} \tag{2.3-37}$$

$\Delta t'$ is given by (2.3-13), $\mathbf{V}_{1,0}$ is given by (2.3-9), $\Delta t''$ is given by (2.3-25), $\mathbf{V}_{2,1}$ is given by (2.3-22), and

$$t = t' + \frac{\left|\mathbf{r}_{1,2}''\right|}{c}, \tag{2.3-38}$$

or

$$\boxed{t' = t - \frac{\left|\mathbf{r}_{1,2}''\right|}{c},} \tag{2.3-39}$$

where $\mathbf{r}_{1,2}''$ is given by (2.3-28). Note that if the bistatic scattering problem shown in Fig. 2.3-1 corresponded to transmission through a space-invariant filter, then the complex frequency response given by (2.3-37) would be a function of the vector spatial difference $\mathbf{r}_2''-\mathbf{r}_0$, which it is *not* [$\mathbf{r}_2''$ is given by (2.3-26)].

In order to evaluate the complex frequency response given by (2.3-37), we must derive solutions for the unit vectors $\hat{n}'_{0,1}$ and $\hat{n}''_{1,2}$, and the ranges $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}''_{1,2}\right|$ in terms of known quantities. Let us begin with $\left|\mathbf{r}'_{0,1}\right|$. Since

$$\left|\mathbf{r}'_{0,1}\right|^2 = \mathbf{r}'_{0,1} \bullet \mathbf{r}'_{0,1}, \tag{2.3-40}$$

substituting (2.3-17) into (2.3-40) yields

$$\left|\mathbf{r}'_{0,1}\right|^2 = (\mathbf{r}_{0,1} + \Delta t' \mathbf{V}_{1,0}) \bullet (\mathbf{r}_{0,1} + \Delta t' \mathbf{V}_{1,0}). \tag{2.3-41}$$

Expanding the right-hand side of (2.3-41) and taking the square root of both sides of the resulting equation yields

$$\boxed{\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\left[1 + \frac{2}{r_{0,1}}(\hat{n}_{0,1} \bullet \mathbf{V}_{1,0})\Delta t' + \left(\frac{V_{1,0}\Delta t'}{r_{0,1}}\right)^2\right]^{1/2},} \tag{2.3-42}$$

or

$$\boxed{\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\left\{1 + 2\frac{V_{1,0}\Delta t'}{r_{0,1}}\left[(\hat{n}_{0,1} \bullet \hat{n}_{V_1}) + \frac{1}{2}\frac{V_{1,0}\Delta t'}{r_{0,1}}\right]\right\}^{1/2}.} \tag{2.3-43}$$

Although (2.3-42) and (2.3-43) are *exact* expressions for the range $\left|\mathbf{r}'_{0,1}\right|$, we cannot compute a value for $\left|\mathbf{r}'_{0,1}\right|$ until we derive an expression for $\Delta t'$ in terms of $\left|\mathbf{r}'_{0,1}\right|$. By referring to Fig. 2.3-3, it can be seen that

$$\boxed{\Delta t' = \frac{\left|\mathbf{r}'_{0,1}\right|}{c}, \qquad t' > t_m.} \tag{2.3-44}$$

Substituting (2.3-44) into (2.3-42) and squaring both sides of the resulting equation yields the following second-order polynomial

$$A\left|\mathbf{r}'_{0,1}\right|^2 - B\left|\mathbf{r}'_{0,1}\right| - C = 0, \tag{2.3-45}$$

with *exact* solution

$$\boxed{\left|\mathbf{r}'_{0,1}\right| = \frac{B \pm \sqrt{B^2 + 4AC}}{2A},} \tag{2.3-46}$$

where

$$A = 1 - \left(\frac{V_{1,0}}{c}\right)^2, \tag{2.3-47}$$

$$B = 2\,r_{0,1}\frac{\hat{n}_{0,1} \bullet \mathbf{V}_{1,0}}{c}, \tag{2.3-48}$$

and

$$C = r_{0,1}^2. \tag{2.3-49}$$

The solution given by (2.3-46) is the *constant* value of range between the point source and the discrete point scatterer when the transmitted acoustic field is *first* incident upon the discrete point scatterer at time instant t' after motion begins at time instant t_m where $t' > t_m$. The decision to use either the plus or minus sign in (2.3-46) is dictated by the fact that range must be positive. Let us solve for $\left|\mathbf{r}_{1,2}''\right|$ next.

Since

$$\left|\mathbf{r}_{1,2}''\right|^2 = \mathbf{r}_{1,2}'' \bullet \mathbf{r}_{1,2}'', \tag{2.3-50}$$

substituting (2.3-29) into (2.3-50) yields

$$\left|\mathbf{r}_{1,2}''\right|^2 = \left[\mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}\right] \bullet \left[\mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}\right]. \tag{2.3-51}$$

Expanding the right-hand side of (2.3-51) and taking the square root of both sides of the resulting equation yields

$$\boxed{\begin{aligned}\left|\mathbf{r}_{1,2}''\right| = r_{1,2}\Bigg\{1 &+ \frac{2}{r_{1,2}}\left[\hat{n}_{1,2} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\Delta t' + \left(\frac{\left|\mathbf{V}_2 - \mathbf{V}_{1,0}\right|\Delta t'}{r_{1,2}}\right)^2 + \frac{2}{r_{1,2}}(\hat{n}_{1,2} \bullet \mathbf{V}_{2,1})\Delta t'' + \left(\frac{V_{2,1}\Delta t''}{r_{1,2}}\right)^2 \\ &+ 2\frac{\Delta t'\Delta t''}{r_{1,2}^2}\left[\mathbf{V}_{2,1} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\Bigg\}^{1/2},\end{aligned}} \tag{2.3-52}$$

or

$$
\boxed{\begin{aligned}
\left|\mathbf{r}''_{1,2}\right| = r_{1,2}\Bigg\{1 &- 2\frac{V_{1,0}\Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2}\bullet\hat{n}_{V_1}) + \frac{\Delta t'}{2r_{1,2}}\Big(V_2(\hat{n}_{V_1}\bullet\hat{n}_{V_2}) - V_{1,0}\Big)\right] \\
&+2\frac{V_2\Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2}\bullet\hat{n}_{V_2}) + \frac{\Delta t'}{2r_{1,2}}\Big(V_2 - V_{1,0}(\hat{n}_{V_1}\bullet\hat{n}_{V_2})\Big)\right] \\
&+2\frac{V_{2,1}\Delta t''}{r_{1,2}}\left[(\hat{n}_{1,2}\bullet\hat{n}_{V_2}) + \frac{V_{2,1}\Delta t''}{2r_{1,2}} + \frac{\Delta t'}{r_{1,2}}\Big(V_2 - V_{1,0}(\hat{n}_{V_1}\bullet\hat{n}_{V_2})\Big)\right]\Bigg\}^{1/2}.
\end{aligned}}
\tag{2.3-53}
$$

Although (2.3-52) and (2.3-53) are *exact* expressions for the range $\left|\mathbf{r}''_{1,2}\right|$, we cannot compute a value for $\left|\mathbf{r}''_{1,2}\right|$ until we derive appropriate expressions for $\Delta t'$, $\Delta t''$, and $\Delta t'\Delta t''$. We will first solve for a constant value for $\left|\mathbf{r}''_{1,2}\right|$ by expressing $\Delta t'$, $\Delta t''$, and $\Delta t'\Delta t''$ in terms of the *known* constant range $\left|\mathbf{r}'_{0,1}\right|$ [see (2.3-46)] and $\left|\mathbf{r}''_{1,2}\right|$. We will then solve for a time-varying $\left|\mathbf{r}''_{1,2}\right|$ by expressing $\Delta t'$, $\Delta t''$, and $\Delta t'\Delta t''$ exclusively in terms of $\left|\mathbf{r}''_{1,2}\right|$. Since (2.3-44) already expresses $\Delta t'$ in terms of $\left|\mathbf{r}'_{0,1}\right|$, we begin by substituting (2.3-38) into (2.3-25) which yields (also see Fig. 2.3-3)

$$
\boxed{\Delta t'' = \frac{\left|\mathbf{r}''_{1,2}\right|}{c}, \qquad t > t' > t_m,}
\tag{2.3-54}
$$

and upon multiplying (2.3-44) by (2.3-54), we obtain

$$
\boxed{\Delta t'\Delta t'' = \frac{\left|\mathbf{r}'_{0,1}\right|}{c}\frac{\left|\mathbf{r}''_{1,2}\right|}{c}, \qquad t > t' > t_m.}
\tag{2.3-55}
$$

The solution for $\left|\mathbf{r}''_{1,2}\right|$ can be obtained by substituting (2.3-54) into (2.3-52) and squaring both sides of the resulting equation. Doing so yields the following second-order polynomial

$$
\mathcal{A}_0\left|\mathbf{r}''_{1,2}\right|^2 - \mathcal{B}_0\left|\mathbf{r}''_{1,2}\right| - \mathcal{C}_0 = 0,
\tag{2.3-56}
$$

with *exact* solution

$$
\boxed{\left|\mathbf{r}''_{1,2}\right| = \frac{\mathcal{B}_0 \pm \sqrt{\mathcal{B}_0^2 + 4\mathcal{A}_0\mathcal{C}_0}}{2\mathcal{A}_0},}
\tag{2.3-57}
$$

where

$$\mathcal{A}_0 = 1 - \frac{A_4}{c^2}, \tag{2.3-58}$$

$$\mathcal{B}_0 = \frac{A_3 + A_5 \Delta t'}{c}, \tag{2.3-59}$$

$$\mathcal{C}_0 = A_2(\Delta t')^2 + A_1 \Delta t' + r_{1,2}^2, \tag{2.3-60}$$

$$A_1 = 2\, r_{1,2}\left[\hat{n}_{1,2} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right], \tag{2.3-61}$$

$$A_2 = \left|\mathbf{V}_2 - \mathbf{V}_{1,0}\right|^2, \tag{2.3-62}$$

$$A_3 = 2\, r_{1,2}\,(\hat{n}_{1,2} \bullet \mathbf{V}_{2,1}), \tag{2.3-63}$$

$$A_4 = V_{2,1}^2, \tag{2.3-64}$$

$$A_5 = 2\left[\mathbf{V}_{2,1} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right], \tag{2.3-65}$$

and

$$\Delta t' = t' - t_m = \frac{\left|\mathbf{r}'_{0,1}\right|}{c}, \qquad t' > t_m, \tag{2.3-66}$$

where $\left|\mathbf{r}'_{0,1}\right|$ is given by (2.3-46). The solution given by (2.3-57) is the *constant* value of range between the discrete point scatterer and the receiver when the scattered acoustic field is *first* incident upon the receiver at time instant t after motion begins at time instant t_m where $t > t' > t_m$. The decision to use either the plus or minus sign in (2.3-57) is dictated by the fact that range must be positive.

As we previously mentioned, we will now solve for a time-varying $\left|\mathbf{r}''_{1,2}\right|$ by expressing $\Delta t'$, $\Delta t''$, and $\Delta t' \Delta t''$ exclusively in terms of $\left|\mathbf{r}''_{1,2}\right|$. We begin by substituting (2.3-39) into (2.3-13) which yields

$$\Delta t' = t - t_m - \frac{\left|\mathbf{r}''_{1,2}\right|}{c}, \qquad t > t' > t_m, \tag{2.3-67}$$

and upon substituing (2.3-7) into (2.3-67), we obtain

$$\boxed{\Delta t' = \Delta t - \frac{\left|\mathbf{r}''_{1,2}\right|}{c}, \qquad t > t' > t_m.} \tag{2.3-68}$$

And upon multiplying (2.3-68) by (2.3-54),

$$\boxed{\Delta t'\Delta t''=\Delta t\frac{\left|\mathbf{r}''_{1,2}\right|}{c}-\left(\frac{\left|\mathbf{r}''_{1,2}\right|}{c}\right)^2, \qquad t>t'>t_m.} \tag{2.3-69}$$

The solution for $\left|\mathbf{r}''_{1,2}\right|$ can be obtained by substituting (2.3-68), (2.3-54), and (2.3-69) into (2.3-52) and squaring both sides of the resulting equation. Doing so yields the following second-order polynomial

$$\mathcal{A}\left|\mathbf{r}''_{1,2}(t)\right|^2+\mathcal{B}(t)\left|\mathbf{r}''_{1,2}(t)\right|-\mathcal{C}(t)=0, \qquad t\geq t_m+\tau, \tag{2.3-70}$$

with *exact* solution

$$\boxed{\left|\mathbf{r}''_{1,2}(t)\right|=\frac{-\mathcal{B}(t)\pm\sqrt{\mathcal{B}^2(t)+4\mathcal{A}\mathcal{C}(t)}}{2\mathcal{A}}, \qquad t\geq t_m+\tau,} \tag{2.3-71}$$

where

$$\mathcal{A}=1-\frac{(A_2+A_4-A_5)}{c^2}, \tag{2.3-72}$$

$$\mathcal{B}(t)=\frac{(2A_2-A_5)\Delta t+(A_1-A_3)}{c}, \tag{2.3-73}$$

$$\mathcal{C}(t)=A_2(\Delta t)^2+A_1\Delta t+r_{1,2}^2, \tag{2.3-74}$$

A_1 through A_5 are given by (2.3-61) through (2.3-65),

$$\Delta t=t-t_m, \qquad t\geq t_m+\tau, \tag{2.3-75}$$

and (see Fig. 2.3-3)

$$\boxed{\tau=\frac{\left|\mathbf{r}'_{0,1}\right|}{c}+\frac{\left|\mathbf{r}''_{1,2}\right|}{c}} \tag{2.3-76}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver after motion begins at time instant t_m) where the *constant* values of range $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}''_{1,2}\right|$ are given by (2.3-46) and (2.3-57), respectively. It is important to note that if (2.3-70) is evaluated at $t=t_m+\tau$, then it can be shown that (2.3-70) can be rewritten in the form of (2.3-56) and, as a result,

$$\left|\mathbf{r}''_{1,2}(t_m+\tau)\right|=\left|\mathbf{r}''_{1,2}\right|, \tag{2.3-77}$$

where $\left|\mathbf{r}''_{1,2}\right|$ is given by (2.3-57).

Now that we have an exact solution for $\left|\mathbf{r}''_{1,2}(t)\right|$ as given by (2.3-71), we can use it to obtain an *exact* solution for the time-varying range $\left|\mathbf{r}'_{0,1}(t)\right|$ as follows. With the use of (2.3-44) and (2.3-68), we can write that

$$\boxed{\left|\mathbf{r}'_{0,1}(t)\right| = c\Delta t - \left|\mathbf{r}''_{1,2}(t)\right|, \qquad t \geq t_m + \tau,} \tag{2.3-78}$$

where Δt is given by (2.3-75), $\left|\mathbf{r}''_{1,2}(t)\right|$ is given by (2.3-71), and τ is given by (2.3-76). Equation (2.3-78) indicates that if we are given a value of Δt, which determines the value of $\left|\mathbf{r}''_{1,2}(t)\right|$, then we can use those two values to compute what $\left|\mathbf{r}'_{0,1}(t)\right|$ must have been - we are working backwards from a value of Δt to a value for $\left|\mathbf{r}''_{1,2}(t)\right|$ to a value for $\left|\mathbf{r}'_{0,1}(t)\right|$. Note that if we evaluate (2.3-78) at $t = t_m + \tau$, then

$$\left|\mathbf{r}'_{0,1}(t_m + \tau)\right| = c\tau - \left|\mathbf{r}''_{1,2}(t_m + \tau)\right|, \tag{2.3-79}$$

and upon substituting (2.3-76) and (2.3-77) into (2.3-79), we obtain

$$\left|\mathbf{r}'_{0,1}(t_m + \tau)\right| = \left|\mathbf{r}'_{0,1}\right|, \tag{2.3-80}$$

where $\left|\mathbf{r}'_{0,1}\right|$ is given by (2.3-46).

Let us next solve for the unit vectors $\hat{n}'_{0,1}$ and $\hat{n}''_{1,2}$ so that we can evaluate the scattering function $g_1\left(f, \hat{n}'_{0,1}, \hat{n}''_{1,2}\right)$ and, hence, the complex frequency response of the ocean. Let us begin with the unit vector $\hat{n}'_{0,1}$. From (2.3-17),

$$\hat{n}'_{0,1} = \frac{1}{\left|\mathbf{r}'_{0,1}\right|}\left(\mathbf{r}_{0,1} + \Delta t' \, \mathbf{V}_{1,0}\right). \tag{2.3-81}$$

Substituting (2.3-66) into (2.3-81) yields

$$\boxed{\hat{n}'_{0,1}(t) = \frac{1}{\left|\mathbf{r}'_{0,1}(t)\right|}\left[\mathbf{r}_{0,1} + \frac{\left|\mathbf{r}'_{0,1}(t)\right|}{c}\mathbf{V}_{1,0}\right], \qquad t \geq t_m + \tau,} \tag{2.3-82}$$

where $\left|\mathbf{r}'_{0,1}(t)\right|$ is given by (2.3-78), $\mathbf{r}_{0,1}$ is given by (2.1-6), $\mathbf{V}_{1,0}$ is given by (2.3-9), and τ is given by (2.3-76). Note that if we evaluate (2.3-82) at $t = t_m + \tau$, then

$$\hat{n}'_{0,1}(t_m+\tau)=\frac{1}{\left|\mathbf{r}'_{0,1}(t_m+\tau)\right|}\left[\mathbf{r}_{0,1}+\frac{\left|\mathbf{r}'_{0,1}(t_m+\tau)\right|}{c}\mathbf{V}_{1,0}\right], \tag{2.3-83}$$

and upon substituting (2.3-80) and (2.3-44) into (2.3-83), we obtain

$$\hat{n}'_{0,1}(t_m+\tau)=\hat{n}'_{0,1}, \tag{2.3-84}$$

where $\hat{n}'_{0,1}$ is given by (2.3-81). And by referring to (2.2-58) and (2.2-59), we can write that

$$\boxed{\theta'_{0,1}(t)=\cos^{-1} w'_{0,1}(t), \qquad t\geq t_m+\tau,} \tag{2.3-85}$$

and

$$\boxed{\psi'_{0,1}(t)=\tan^{-1}\left(\frac{v'_{0,1}(t)}{u'_{0,1}(t)}\right), \qquad t\geq t_m+\tau,} \tag{2.3-86}$$

where $u'_{0,1}(t)$, $v'_{0,1}(t)$, and $w'_{0,1}(t)$ are the dimensionless, time-varying direction cosines with respect to the X, Y, and Z axes, respectively, associated with the time-varying unit vector $\hat{n}'_{0,1}(t)$ given by (2.3-82), and τ is given by (2.3-76). Equations (2.3-85) and (2.3-86) are the *angles of incidence* at the discrete point scatterer.

From (2.3-28),

$$\hat{n}''_{1,2}=\frac{1}{\left|\mathbf{r}''_{1,2}\right|}\left[\mathbf{r}_{1,2}+\Delta t'(\mathbf{V}_2-\mathbf{V}_{1,0})+\Delta t''\mathbf{V}_{2,1}\right]. \tag{2.3-87}$$

Substituting (2.3-68) and (2.3-54) into (2.3-87) yields

$$\boxed{\hat{n}''_{1,2}(t)=\frac{1}{\left|\mathbf{r}''_{1,2}(t)\right|}\left\{\mathbf{r}_{1,2}+\left[\Delta t-\frac{\left|\mathbf{r}''_{1,2}(t)\right|}{c}\right](\mathbf{V}_2-\mathbf{V}_{1,0})+\frac{\left|\mathbf{r}''_{1,2}(t)\right|}{c}\mathbf{V}_{2,1}\right\}, \qquad t\geq t_m+\tau,} \tag{2.3-88}$$

where $\left|\mathbf{r}''_{1,2}(t)\right|$ is given by (2.3-71), $\mathbf{r}_{1,2}$ is given by (2.1-14), Δt is given by (2.3-75), $\mathbf{V}_{1,0}$ is given by (2.3-9), $\mathbf{V}_{2,1}$ is given by (2.3-22), and τ is given by (2.3-76). Note that if we evaluate (2.3-88) at $t=t_m+\tau$, then

$$\hat{n}''_{1,2}(t_m+\tau)=\frac{1}{\left|\mathbf{r}''_{1,2}(t_m+\tau)\right|}\left\{\mathbf{r}_{1,2}+\left[\tau-\frac{\left|\mathbf{r}''_{1,2}(t_m+\tau)\right|}{c}\right](\mathbf{V}_2-\mathbf{V}_{1,0})+\frac{\left|\mathbf{r}''_{1,2}(t_m+\tau)\right|}{c}\mathbf{V}_{2,1}\right\}, \tag{2.3-89}$$

and upon substituting (2.3-76), (2.3-77), (2.3-44), and (2.3-54) into (2.3-89), we obtain

$$\hat{n}''_{1,2}(t_m + \tau) = \hat{n}''_{1,2}, \tag{2.3-90}$$

where $\hat{n}''_{1,2}$ is given by (2.3-87). And by referring to (2.2-64) and (2.2-65), we can write that

$$\boxed{\theta''_{1,2}(t) = \cos^{-1} w''_{1,2}(t), \qquad t \geq t_m + \tau,} \tag{2.3-91}$$

and

$$\boxed{\psi''_{1,2}(t) = \tan^{-1}\left(\frac{v''_{1,2}(t)}{u''_{1,2}(t)}\right), \qquad t \geq t_m + \tau,} \tag{2.3-92}$$

where $u''_{1,2}(t)$, $v''_{1,2}(t)$, and $w''_{1,2}(t)$ are the dimensionless, time-varying direction cosines with respect to the X, Y, and Z axes, respectively, associated with the time-varying unit vector $\hat{n}''_{1,2}(t)$ given by (2.3-88), and τ is given by (2.3-76). Equations (2.3-91) and (2.3-92) are the *angles of scatter* at the receiver.

Note that since the unit vectors $\hat{n}'_{0,1}(t)$ and $\hat{n}''_{1,2}(t)$ given by (2.3-82) and (2.3-88), respectively, are functions of time, the scattering function is also a function of time, that is,

$$g_1\left(f, \hat{n}'_{0,1}, \hat{n}''_{1,2}\right) = g_1\left(f, \hat{n}'_{0,1}(t), \hat{n}''_{1,2}(t)\right), \qquad t \geq t_m + \tau. \tag{2.3-93}$$

Let us next derive an expression for the time-varying position vector to the receiver when the scattered acoustic field is incident upon the receiver. We begin by substituting (2.3-20) into (2.3-26). Doing so yields

$$\mathbf{r}''_2 = \mathbf{r}_2 + \Delta t' \, \mathbf{V}_2 + \Delta t'' \, \mathbf{V}_{2,1}, \tag{2.3-94}$$

which is the position vector from the origin to the receiver when the scattered acoustic field is *first* incident upon the receiver at time instant t after motion begins at time instant t_m where $t > t' > t_m$. And upon substituting (2.3-68) and (2.3-54) into (2.3-94), we obtain

$$\boxed{\mathbf{r}''_2(t) = \mathbf{r}_2 + \left[\Delta t - \frac{\left|\mathbf{r}''_{1,2}(t)\right|}{c}\right]\mathbf{V}_2 + \frac{\left|\mathbf{r}''_{1,2}(t)\right|}{c}\mathbf{V}_{2,1}, \qquad t \geq t_m + \tau,} \tag{2.3-95}$$

where Δt is given by (2.3-75), $\left|\mathbf{r}''_{1,2}(t)\right|$ is given by (2.3-71), $\mathbf{V}_{2,1}$ is given by (2.3-22), and τ is given by (2.3-76). Note that if we evaluate (2.3-95) at $t = t_m + \tau$, then

$$\mathbf{r}''_2(t_m + \tau) = \mathbf{r}_2 + \left[\tau - \frac{\left|\mathbf{r}''_{1,2}(t_m + \tau)\right|}{c}\right]\mathbf{V}_2 + \frac{\left|\mathbf{r}''_{1,2}(t_m + \tau)\right|}{c}\mathbf{V}_{2,1}, \tag{2.3-96}$$

and upon substituting (2.3-76) and (2.3-77) into (2.3-96), we obtain

$$\mathbf{r}_2''(t_m+\tau)=\mathbf{r}_2+\frac{\left|\mathbf{r}_{0,1}'\right|}{c}\mathbf{V}_2+\frac{\left|\mathbf{r}_{1,2}''\right|}{c}\mathbf{V}_{2,1}, \tag{2.3-97}$$

where $\left|\mathbf{r}_{0,1}'\right|$ and $\left|\mathbf{r}_{1,2}''\right|$ are given by (2.3-46) and (2.3-57), respectively. If we further substitute (2.3-44) and (2.3-54) into (2.3-97), then

$$\mathbf{r}_2''(t_m+\tau)=\mathbf{r}_2'', \tag{2.3-98}$$

where $\mathbf{r}_2''$ is given by (2.3-94).

With the use of (2.3-36) and (2.3-37), and by replacing the real wavenumber k in (2.3-37) with the complex wavenumber K given by (2.1-20), we can *summarize* our results as follows: for the bistatic scattering problem shown in Fig. 2.3-1, the time-harmonic velocity potential in squared-meters per second incident upon the receiver at $\mathbf{r}_2''(t)=\left(x_2''(t),y_2''(t),z_2''(t)\right)$, due to a moving time-harmonic point source initially at $\mathbf{r}_0=(x_0,y_0,z_0)$ and a moving discrete point scatterer initially at $\mathbf{r}_1=(x_1,y_1,z_1)$, is given by

$$\boxed{y_M\left(t,\mathbf{r}_2''(t)\right)=S_0H_M\left(t,\mathbf{r}_2''(t)\middle|f,\mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t\geq t_m+\tau,} \tag{2.3-99}$$

where $\mathbf{r}_2''(t)$ is given by (2.3-95), S_0 is the source strength in cubic meters per second,

$$\boxed{H_M\left(t,\mathbf{r}_2''(t)\middle|f,\mathbf{r}_0\right)=g_1\left(f,\hat{n}_{0,1}'(t),\hat{n}_{1,2}''(t)\right)\frac{\exp\left[-\alpha(f)\left(\left|\mathbf{r}_{0,1}'(t)\right|+\left|\mathbf{r}_{1,2}''(t)\right|\right)\right]}{16\pi^2\left|\mathbf{r}_{0,1}'(t)\right|\left|\mathbf{r}_{1,2}''(t)\right|}\exp\left[-jk\left(\left|\mathbf{r}_{0,1}'(t)\right|+\left|\mathbf{r}_{1,2}''(t)\right|\right)\right]} \tag{2.3-100}$$

is the time-variant, space-variant, complex frequency response of the ocean at frequency f hertz,

$$\boxed{g_1\left(f,\hat{n}_{0,1}'(t),\hat{n}_{1,2}''(t)\right)=g_1\left(f,\theta_{0,1}'(t),\psi_{0,1}'(t),\theta_{1,2}''(t),\psi_{1,2}''(t)\right)} \tag{2.3-101}$$

is the scattering function of the discrete point scatterer in meters, the angles of incidence $\theta_{0,1}'(t)$ and $\psi_{0,1}'(t)$ are given by (2.3-85) and (2.3-86), respectively, the angles of scatter $\theta_{1,2}''(t)$ and $\psi_{1,2}''(t)$ are given by (2.3-91) and (2.3-92), respectively, $\alpha(f)$ is the real, frequency-dependent, attenuation coefficient in nepers per meter, $\left|\mathbf{r}_{0,1}'(t)\right|$ is given by (2.3-78), $\left|\mathbf{r}_{1,2}''(t)\right|$ is given by (2.3-71), the real wavenumber k in radians per meter is given by (2.1-5), and the time delay τ in seconds is given by (2.3-76).

Let us next relate the scattering function, the differential scattering cross section, and target strength of the discrete point scatterer. The *target strength* (TS) is defined as follows [6]:

$$\boxed{\mathrm{TS} \triangleq 10\log_{10}\left[\frac{\sigma_d\left(f,\hat{n}'_{0,1}(t),\hat{n}''_{1,2}(t)\right)}{A_{\mathrm{ref}}}\right] \mathrm{dB\ re}\ A_{\mathrm{ref}}, \qquad t \geq t_m + \tau,} \tag{2.3-102}$$

where [5, 6]

$$\boxed{\sigma_d\left(f,\hat{n}'_{0,1}(t),\hat{n}''_{1,2}(t)\right) \triangleq \lim_{r''_{1,2}(t)\to\infty}\left[\frac{[r''_{1,2}(t)]^2 I_{avg_s}\left(\mathbf{r}''_2(t)\right)}{I_{avg_i}\left(\mathbf{r}'_1(t)\right)}\right] = \frac{\left|g_1\left(f,\hat{n}'_{0,1}(t),\hat{n}''_{1,2}(t)\right)\right|^2}{(4\pi)^2}, \qquad t \geq t_m + \tau,}$$

(2.3-103)

is the *differential scattering cross section* with units of squared meters, $I_{avg_i}\left(\mathbf{r}'_1(t)\right)$ and $I_{avg_s}\left(\mathbf{r}''_2(t)\right)$ are the *time-average*, *incident* and *scattered intensities*, respectively, with units of watts per squared meter, $g_1\left(f,\hat{n}'_{0,1}(t),\hat{n}''_{1,2}(t)\right)$ is the scattering function of the discrete point scatterer with units of meters, A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to $1\ \mathrm{m}^2$, and the time delay τ in seconds is given by (2.3-76).

Example 2.3-1 Simulation of Sections 2.1 and 2.2

In this example we will show that the exact results derived in this section will reduce to the exact results derived in Sections 2.1 and 2.2 when appropriate values are used for the various parameters. Let us begin with the problem discussed in Section 2.2.

Simulation of Section 2.2

In Section 2.2, only the discrete point scatterer was in motion with constant velocity vector $\mathbf{V}_1$. In order to simulate this problem, we set the constant velocity vector of the transmitter $\mathbf{V}_0 = \mathbf{0}$ and the constant velocity vector of the receiver $\mathbf{V}_2 = \mathbf{0}$. As a result [see (2.3-9)],

$$\mathbf{V}_{1,0} = \mathbf{V}_1, \tag{2.3-104}$$

$$\begin{aligned} V_{1,0} &= \left|\mathbf{V}_{1,0}\right| = \left|\mathbf{V}_1\right| \\ &= V_1, \end{aligned} \tag{2.3-105}$$

$$\mathbf{V}_{2,1} = \mathbf{0} \tag{2.3-106}$$

since $\mathbf{V}_{2,1}$ is undefined when $\mathbf{V}_2 = \mathbf{0}$ [see (2.3-22)], and

$$V_{2,1} = \left|\mathbf{V}_{2,1}\right| = 0. \tag{2.3-107}$$

Substituting (2.3-105) into (2.3-47), (2.3-104) into (2.3-48), and repeating (2.3-49) for convenience yields

$$A = 1 - \left(\frac{V_1}{c}\right)^2, \tag{2.3-108}$$

$$B = 2\,r_{0,1}\frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c}, \tag{2.3-109}$$

and

$$C = r_{0,1}^2. \tag{2.3-110}$$

Equations (2.3-108) through (2.3-110) are *identical* with (2.2-33) through (2.2-35). Therefore, with the use of (2.3-108) through (2.3-110), the exact solution for the constant value of range $\left|\mathbf{r}'_{0,1}\right|$ given by (2.3-46) reduces to the exact solution given by (2.2-32). Recall that $\left|\mathbf{r}'_{0,1}\right|$ is the constant value of range between the point source and the discrete point scatterer when the transmitted acoustic field is *first* incident upon the discrete point scatterer at time instant t' after motion begins at time instant t_m where $t' > t_m$.

Substituting $\mathbf{V}_2 = \mathbf{0}$ and (2.3-104) through (2.3-107) into (2.3-61) through (2.3-65) yields

$$A_1 = -2\,r_{1,2}\,(\hat{n}_{1,2} \bullet \mathbf{V}_1), \tag{2.3-111}$$

$$A_2 = V_1^2, \tag{2.3-112}$$

$$A_3 = 0, \tag{2.3-113}$$

$$A_4 = 0, \tag{2.3-114}$$

and

$$A_5 = 0. \tag{2.3-115}$$

Substituting (2.3-113) through (2.3-115) into (2.3-58) and (2.3-59), and repeating (2.3-60) for convenience yields

$$\mathcal{A}_0 = 1, \tag{2.3-116}$$

$$\mathcal{B}_0 = 0, \tag{2.3-117}$$

and

$$\mathcal{C}_0 = A_2(\Delta t')^2 + A_1\Delta t' + r_{1,2}^2. \tag{2.3-118}$$

Substituting (2.3-116) through (2.3-118) into (2.3-57) yields

$$\left|\mathbf{r}''_{1,2}\right| = \sqrt{A_2(\Delta t')^2 + A_1\Delta t' + r_{1,2}^2}\,, \tag{2.3-119}$$

and upon substituting (2.3-66), (2.3-111), and (2.3-112) into (2.3-119), we finally obtain

$$\left|\mathbf{r}_{1,2}''\right| = \left|\mathbf{r}_{1,2}'\right|, \tag{2.3-120}$$

where $\left|\mathbf{r}_{1,2}'\right|$ is given by (2.2-40). Therefore, the exact solution for the constant value of range $\left|\mathbf{r}_{1,2}''\right|$ given by (2.3-57) reduces to the exact solution for the constant value of range $\left|\mathbf{r}_{1,2}'\right|$ given by (2.2-40). Recall that both $\left|\mathbf{r}_{1,2}''\right|$ and $\left|\mathbf{r}_{1,2}'\right|$ are the constant values of range between the discrete point scatterer and the receiver when the scattered acoustic field is *first* incident upon the receiver at time instant t after motion begins at time instant t_m where $t > t' > t_m$.

Substituting (2.3-111) through (2.3-115) into (2.3-72) through (2.3-74) yields

$$\mathcal{A} = 1 - \left(\frac{V_1}{c}\right)^2, \tag{2.3-121}$$

$$\mathcal{B}(t) = 2\frac{V_1^2}{c}\Delta t - 2\,r_{1,2}\frac{\hat{n}_{1,2} \bullet \mathbf{V}_1}{c}, \tag{2.3-122}$$

and

$$\mathcal{C}(t) = V_1^2(\Delta t)^2 - 2\,r_{1,2}(\hat{n}_{1,2} \bullet \mathbf{V}_1)\Delta t + r_{1,2}^2, \tag{2.3-123}$$

where Δt is given by (2.3-75). And because of (2.3-120), Δt given by (2.3-75) is *identical* to Δt given by (2.2-48) since the time delay τ given by (2.3-76) is *identical* to the time delay τ given by (2.2-49). As a result, (2.3-121) through (2.3-123) are *identical* with (2.2-45) through (2.2-47). Therefore, with the use of (2.3-121) through (2.3-123), the exact solution for the time-varying range $\left|\mathbf{r}_{1,2}''(t)\right|$ given by (2.3-71) reduces to the exact solution for the time-varying range $\left|\mathbf{r}_{1,2}'(t)\right|$ given by (2.2-44), that is,

$$\left|\mathbf{r}_{1,2}''(t)\right| = \left|\mathbf{r}_{1,2}'(t)\right|. \tag{2.3-124}$$

With the use of (2.3-124), the exact solution for the time-varying range $\left|\mathbf{r}_{0,1}'(t)\right|$ given by (2.3-78) reduces to the exact solution given by (2.2-51). With the use of (2.3-104), the exact solution for the time-varying unit vector $\hat{n}_{0,1}'(t)$ given by (2.3-82) reduces to the exact solution given by (2.2-55). With the use of $\mathbf{V}_2 = \mathbf{0}$, (2.3-104), (2.3-106), and (2.3-124), the exact solution for the time-varying unit vector $\hat{n}_{1,2}''(t)$ given by (2.3-88) reduces to the exact solution for the time-varying unit vector $\hat{n}_{1,2}'(t)$ given by (2.2-61), that is,

$$\hat{n}_{1,2}''(t) = \hat{n}_{1,2}'(t). \tag{2.3-125}$$

And finally, substituting $\mathbf{V}_2 = \mathbf{0}$ and (2.3-106) into (2.3-95) yields

$$\mathbf{r}_2''(t) = \mathbf{r}_2. \tag{2.3-126}$$

Let us next consider the problem discussed in Section 2.1.

Simulation of Section 2.1

In Section 2.1 none of the platforms were in motion. In order to simulate this problem, all we need to do for the most part is to set $\mathbf{V}_1 = \mathbf{0}$ and $V_1 = 0$ into the equations already derived in this example.

Substituting $V_1 = 0$ into (2.3-108) and $\mathbf{V}_1 = \mathbf{0}$ into (2.3-109), and repeating (2.3-110) for convenience yields

$$A = 1, \tag{2.3-127}$$

$$B = 0, \tag{2.3-128}$$

and

$$C = r_{0,1}^2. \tag{2.3-129}$$

Substituting (2.3-127) through (2.3-129) into (2.3-46) yields

$$\left|\mathbf{r}'_{0,1}\right| = \left|\mathbf{r}_{0,1}\right| = r_{0,1} \tag{2.3-130}$$

as the constant value of range between the point source and the discrete point scatterer [see (2.1-26)].

Substituting $\mathbf{V}_1 = \mathbf{0}$ into (2.3-111) and $V_1 = 0$ into (2.3-112), and repeating (2.3-113) through (2.3-115) for convenience yields

$$A_1 = 0, \tag{2.3-131}$$

$$A_2 = 0, \tag{2.3-132}$$

$$A_3 = 0, \tag{2.3-133}$$

$$A_4 = 0, \tag{2.3-134}$$

and

$$A_5 = 0. \tag{2.3-135}$$

Substituting (2.3-131) and (2.3-132) into (2.3-118) and (2.3-119), and repeating (2.3-116) and (2.3-117) for convenience yields

$$\mathcal{A}_0 = 1, \tag{2.3-136}$$

$$\mathcal{B}_0 = 0, \tag{2.3-137}$$

$$\mathcal{C}_0 = r_{1,2}^2, \tag{2.3-138}$$

and

$$\left|\mathbf{r}''_{1,2}\right| = \left|\mathbf{r}_{1,2}\right| = r_{1,2} \tag{2.3-139}$$

as the constant value of range between the discrete point scatterer and the receiver [see (2.1-27)]. Note that with the use of (2.3-130) and (2.3-139), the time delay τ given by (2.3-76) reduces to the time delay τ given by (2.1-25).

Substituting $\mathbf{V}_1 = \mathbf{0}$ and $V_1 = 0$ into (2.3-121) through (2.3-123) yields

$$\mathcal{A} = 1, \tag{2.3-140}$$

$$\mathcal{B}(t) = 0, \tag{2.3-141}$$

and

$$\mathcal{C}(t) = r_{1,2}^2. \tag{2.3-142}$$

Substituting (2.3-140) through (2.3-142) into (2.3-71) yields

$$\left|\mathbf{r}''_{1,2}(t)\right| = \left|\mathbf{r}_{1,2}\right| = r_{1,2}. \tag{2.3-143}$$

Equation (2.3-143) indicates that the range between the discrete point scatterer and the receiver does not change value as a function of time - it is equal to the constant value $r_{1,2}$ - which makes physical sense since none of the platforms are in motion.

When none of the platforms are in motion, the time delay between the transmitter, discrete point scatterer, and receiver is *constant*, that is, $\Delta t = \tau$, where τ is given by (2.3-76). Therefore, with the use of (2.3-76), (2.3-130), and (2.3-139),

$$\Delta t = \tau = \frac{\left|\mathbf{r}_{0,1}\right|}{c} + \frac{\left|\mathbf{r}_{1,2}\right|}{c}. \tag{2.3-144}$$

Substituting (2.3-143) and (2.3-144) into (2.3-78) yields

$$\left|\mathbf{r}'_{0,1}(t)\right| = \left|\mathbf{r}_{0,1}\right| = r_{0,1}. \tag{2.3-145}$$

Equation (2.3-145) indicates that the range between the transmitter and the discrete point scatterer does not change value as a function of time - it is equal to the constant value $r_{0,1}$ - which makes physical sense since none of the platforms are in motion.

Substituting $\mathbf{V}_1 = \mathbf{0}$ into (2.3-104) yields

$$\mathbf{V}_{1,0} = \mathbf{0}. \tag{2.3-146}$$

With the use of (2.3-145) and (2.3-146), the exact solution for the time-varying unit vector $\hat{n}'_{0,1}(t)$ given by (2.3-82) reduces to the exact solution for the constant unit vector $\hat{n}_{0,1}$ given by (2.1-18), that is,

$$\hat{n}'_{0,1}(t) = \hat{n}_{0,1} = \frac{\mathbf{r}_{0,1}}{\left|\mathbf{r}_{0,1}\right|}. \tag{2.3-147}$$

And finally, with the use of $\mathbf{V}_2 = \mathbf{0}$, (2.3-106), (2.3-143), and (2.3-146), the exact solution for the time-varying unit vector $\hat{n}''_{1,2}(t)$ given by (2.3-88) reduces to the exact solution for the constant unit vector $\hat{n}_{1,2}$ given by (2.1-19), that is,

$$\hat{n}''_{1,2}(t) = \hat{n}_{1,2} = \frac{\mathbf{r}_{1,2}}{\left|\mathbf{r}_{1,2}\right|}. \tag{2.3-148}$$

Example 2.3-2 Monostatic Scattering Geometry

In this example *all* three platforms are in motion with constant velocity vectors $\mathbf{V}_0$, $\mathbf{V}_1$, and $\mathbf{V}_2$, but the scattering geometry is monostatic versus bistatic. For a *monostatic* (*backscatter*) scattering geometry, both the transmitter and receiver are located at the same position, that is,

$$\mathbf{r}_2 = \mathbf{r}_0. \tag{2.3-149}$$

Substituting (2.3-149) into (2.1-14) yields

$$\mathbf{r}_{1,2} = -\mathbf{r}_{0,1}, \tag{2.3-150}$$

where $\mathbf{r}_{0,1}$ is given by (2.1-6). Therefore,

$$r_{1,2} = r_{0,1} \tag{2.3-151}$$

and

$$\hat{n}_{1,2} = -\hat{n}_{0,1}. \tag{2.3-152}$$

And since both the transmitter and receiver are on the same platform,

$$\mathbf{V}_2 = \mathbf{V}_0, \tag{2.3-153}$$

$$V_2 = V_0, \tag{2.3-154}$$

and

$$\hat{n}_{V_2} = \hat{n}_{V_0}. \tag{2.3-155}$$

Substituting (2.3-153) through (2.3-155) into (2.3-22) yields the following expression for the velocity vector of the receiver *relative* to the velocity vector of the discrete point scatterer *in the direction of the velocity vector of the receiver* $\hat{n}_{V_2} = \hat{n}_{V_0}$:

$$\mathbf{V}_{2,1} = \mathbf{V}_0 - (\hat{n}_{V_0} \bullet \mathbf{V}_1)\hat{n}_{V_0} = \left[V_0 - V_1(\hat{n}_{V_0} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_0}. \tag{2.3-156}$$

The time variant, space variant, complex frequency response of the ocean for a monostatic scattering geometry with all three platforms in motion is given by (2.3-100) in conjunction with (2.3-149) through (2.3-156).

Example 2.3-3 Synthetic Aperture Sonar (SAS)

In this example we will consider a *bistatic* scattering geometry consistent with a synthetic aperture sonar (SAS) trying to image a nonmoving target on the ocean bottom. Although the transmitter and receiver are on the same platform, they are *not* located at the same position, that is,

$$\mathbf{r}_2 \neq \mathbf{r}_0. \tag{2.3-157}$$

However, (2.3-153) through (2.3-156) are applicable in this example.

As was mentioned, the discrete point scatterer (target) is *not* in motion. Therefore,

$$\mathbf{V}_1 = \mathbf{0}, \tag{2.3-158}$$

and, as a result, the relative velocity vector $\mathbf{V}_{1,0}$ given by (2.3-9) is *undefined*. Therefore,

$$\mathbf{V}_{1,0} = \mathbf{0} \tag{2.3-159}$$

and

$$V_{1,0} = \left|\mathbf{V}_{1,0}\right| = 0. \tag{2.3-160}$$

And, upon substituting (2.3-158) into (2.3-156), we obtain

$$\mathbf{V}_{2,1} = \mathbf{V}_0 \tag{2.3-161}$$

and

$$\begin{aligned} V_{2,1} &= \left|\mathbf{V}_{2,1}\right| = \left|\mathbf{V}_0\right| \\ &= V_0. \end{aligned} \tag{2.3-162}$$

Let us solve for the constant value of range $\left|\mathbf{r}'_{0,1}\right|$ next. Substituting (2.3-160) into (2.3-47) and (2.3-159) into (2.3-48), and repeating (2.3-49) for convenience yields

$$A = 1, \tag{2.3-163}$$

$$B = 0, \tag{2.3-164}$$

and

$$C = r_{0,1}^2. \tag{2.3-165}$$

Substituting (2.3-163) through (2.3-165) into (2.3-46) yields

$$\boxed{\left|\mathbf{r}'_{0,1}\right| = r_{0,1}} \tag{2.3-166}$$

as the *constant* value of range between the point source and the discrete point scatterer when the transmitted acoustic field is *first* incident upon the discrete point scatterer at time instant t' after motion begins at time instant t_m where $t' > t_m$. The range $r_{0,1}$ is given by (2.1-26).

Now let us solve for the constant value of range $|\mathbf{r}''_{1,2}|$. Substituting (2.3-153) and (2.3-159) into (2.3-61); (2.3-153), (2.3-154), and (2.3-159) into (2.3-62); (2.3-161) into (2.3-63); (2.3-162) into (2.3-64); (2.3-153), (2.3-154), (2.3-159), and (2.3-161) into (2.3-65); and (2.3-166) into (2.3-66) yields

$$A_1 = 2 r_{1,2} (\hat{n}_{1,2} \bullet \mathbf{V}_0), \tag{2.3-167}$$

$$A_2 = V_0^2, \tag{2.3-168}$$

$$A_3 = 2 r_{1,2} (\hat{n}_{1,2} \bullet \mathbf{V}_0), \tag{2.3-169}$$

$$A_4 = V_0^2, \tag{2.3-170}$$

$$A_5 = 2V_0^2, \tag{2.3-171}$$

and

$$\Delta t' = t' - t_m = \frac{r_{0,1}}{c}, \qquad t' > t_m. \tag{2.3-172}$$

With the use of (2.3-57) and substituting (2.3-170) into (2.3-58); (2.3-169), (2.3-171), and (2.3-172) into (2.3-59); and (2.3-167), (2.3-168), and (2.3-172) into (2.3-60), we obtain

$$\boxed{|\mathbf{r}''_{1,2}| = \frac{\mathcal{B}_0 \pm \sqrt{\mathcal{B}_0^2 + 4\mathcal{A}_0\mathcal{C}_0}}{2\mathcal{A}_0},} \tag{2.3-173}$$

where

$$\mathcal{A}_0 = 1 - \frac{V_0^2}{c^2}, \tag{2.3-174}$$

$$\mathcal{B}_0 = 2 r_{1,2} \frac{(\hat{n}_{1,2} \bullet \mathbf{V}_0)}{c} + 2 r_{0,1} \frac{V_0^2}{c^2}, \tag{2.3-175}$$

and

$$\mathcal{C}_0 = r_{0,1}^2 \frac{V_0^2}{c^2} + 2 r_{0,1} r_{1,2} \frac{(\hat{n}_{1,2} \bullet \mathbf{V}_0)}{c} + r_{1,2}^2. \tag{2.3-176}$$

The ranges $r_{0,1}$ and $r_{1,2}$ are given by (2.1-26) and (2.1-27), respectively. The solution given by (2.3-173) is the *constant* value of range between the discrete point scatterer and the receiver when the scattered acoustic field is *first* incident upon the receiver at time instant t after motion begins at time

instant t_m where $t > t' > t_m$. The decision to use either the plus or minus sign in (2.3-173) is dictated by the fact that range must be positive.

Let us solve for the time-varying range $\left|\mathbf{r}''_{1,2}(t)\right|$ next. Substituting (2.3-168), (2.3-170), and (2.3-171) into (2.3-72); (2.3-167) through (2.3-169) and (2.3-171) into (2.3-73); and (2.3-167) and (2.3-168) into (2.3-74) yields

$$\mathcal{A} = 1, \tag{2.3-177}$$

$$\mathcal{B}(t) = 0, \tag{2.3-178}$$

and

$$\mathcal{C}(t) = V_0^2(\Delta t)^2 + 2\,r_{1,2}(\hat{n}_{1,2} \bullet \mathbf{V}_0)\Delta t + r_{1,2}^2, \tag{2.3-179}$$

where [see (2.3-75) and (2.3-76)]

$$\boxed{\Delta t = t - t_m, \qquad t \geq t_m + \tau,} \tag{2.3-180}$$

and

$$\boxed{\tau = \frac{\left|\mathbf{r}'_{0,1}\right|}{c} + \frac{\left|\mathbf{r}''_{1,2}\right|}{c}} \tag{2.3-181}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver after motion begins at time instant t_m) where the *constant* values of range $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}''_{1,2}\right|$ are given by (2.3-166) and (2.3-173), respectively. Substituting (2.3-177) through (2.3-179) into (2.3-71) yields

$$\boxed{\left|\mathbf{r}''_{1,2}(t)\right| = \left[r_{1,2}^2 + 2\,r_{1,2}(\hat{n}_{1,2} \bullet \mathbf{V}_0)\Delta t + V_0^2(\Delta t)^2\right]^{1/2}, \qquad t \geq t_m + \tau.} \tag{2.3-182}$$

Now let us solve for the time-varying range $\left|\mathbf{r}'_{0,1}(t)\right|$. With the use of (2.3-78), we can write that

$$\boxed{\left|\mathbf{r}'_{0,1}(t)\right| = c\Delta t - \left|\mathbf{r}''_{1,2}(t)\right|, \qquad t \geq t_m + \tau,} \tag{2.3-183}$$

where Δt is given by (2.3-180), $\left|\mathbf{r}''_{1,2}(t)\right|$ is given by (2.3-182), and τ is given by (2.3-181).

Let us solve for the time-varying unit vector $\hat{n}'_{0,1}(t)$ next. If we substitute (2.3-159) into (2.3-82), then

$$\hat{n}'_{0,1}(t) = \frac{\mathbf{r}_{0,1}}{\left|\mathbf{r}'_{0,1}(t)\right|}, \qquad t \geq t_m + \tau. \tag{2.3-184}$$

Equation (2.3-184) does *not* make physical sense because we need the time-varying vector $\mathbf{r}'_{0,1}(t)$ in the numerator and not the constant vector $\mathbf{r}_{0,1}$. Therefore, we need to derive an equation for $\mathbf{r}'_{0,1}(t)$. Before we begin the derivation, let us note that although (2.3-184) does not make physical sense, if we substitute (2.3-159) into (2.3-17), then

$$\mathbf{r}'_{0,1} = \mathbf{r}_{0,1} \tag{2.3-185}$$

and, as a result,

$$\left|\mathbf{r}'_{0,1}\right| = \left|\mathbf{r}_{0,1}\right| = r_{0,1}. \tag{2.3-186}$$

Equation (2.3-186) *does* make physical sense and it agrees with (2.3-166).

We begin the derivation of the equation for the time-varying vector $\mathbf{r}'_{0,1}(t)$ by substituting (2.1-6) into (2.3-185) yielding

$$\mathbf{r}'_{0,1} = \mathbf{r}_1 - \mathbf{r}_0. \tag{2.3-187}$$

We then generalize (2.3-187) as follows:

$$\mathbf{r}'_{0,1}(t) = \mathcal{R}_1(t-\tau) - \mathcal{R}_0(t-\tau), \qquad t \geq t_m + \tau, \tag{2.3-188}$$

where $\mathcal{R}_0(t)$ is the time-varying position vector from the origin to the transmitter given by (2.3-4), and $\mathcal{R}_1(t)$ is the time-varying position vector from the origin to the discrete point scatterer given by (2.3-5). Substituting (2.3-4), (2.3-5), and (2.3-158) into (2.3-188) yields

$$\boxed{\mathbf{r}'_{0,1}(t) = \mathbf{r}_{0,1} - [t-(t_m+\tau)]\mathbf{V}_0, \qquad t \geq t_m + \tau,} \tag{2.3-189}$$

where $\mathbf{r}_{0,1}$ is given by (2.1-6) and τ is given by (2.3-181). Note that if we evaluate (2.3-189) at $t = t_m + \tau$, then

$$\mathbf{r}'_{0,1}(t_m+\tau) = \mathbf{r}'_{0,1}, \tag{2.3-190}$$

where $\mathbf{r}'_{0,1}$ is given by (2.3-185). Therefore,

$$\left|\mathbf{r}'_{0,1}(t_m+\tau)\right| = \left|\mathbf{r}'_{0,1}\right|, \tag{2.3-191}$$

where $\left|\mathbf{r}'_{0,1}\right|$ is given by either (2.3-186) or (2.3-166). Equation (2.3-191) agrees with the general result given by (2.3-80).

As a result of the above analysis, the time-varying unit vector $\hat{n}'_{0,1}(t)$ is given by

$$\boxed{\hat{n}'_{0,1}(t) = \frac{\mathbf{r}'_{0,1}(t)}{\left|\mathbf{r}'_{0,1}(t)\right|}, \qquad t \ge t_m + \tau,} \tag{2.3-192}$$

where $\mathbf{r}'_{0,1}(t)$ is given by (2.3-189) and τ is given by (2.3-181). And by referring to (2.3-85) and (2.3-86), we can write that

$$\boxed{\theta'_{0,1}(t) = \cos^{-1} w'_{0,1}(t), \qquad t \ge t_m + \tau,} \tag{2.3-193}$$

and

$$\boxed{\psi'_{0,1}(t) = \tan^{-1}\left(\frac{v'_{0,1}(t)}{u'_{0,1}(t)}\right), \qquad t \ge t_m + \tau,} \tag{2.3-194}$$

where $u'_{0,1}(t)$, $v'_{0,1}(t)$, and $w'_{0,1}(t)$ are the dimensionless, time-varying direction cosines with respect to the X, Y, and Z axes, respectively, associated with the time-varying unit vector $\hat{n}'_{0,1}(t)$ given by (2.3-192), and τ is given by (2.3-181). Equations (2.3-193) and (2.3-194) are the *angles of incidence* at the discrete point scatterer that are to be used to evaluate the scattering function of the discrete point scatterer given by (2.3-101).

Let us solve for the time-varying unit vector $\hat{n}''_{1,2}(t)$ next. If we substitute (2.3-153), (2.3-159), and (2.3-161) into (2.3-88), then

$$\boxed{\hat{n}''_{1,2}(t) = \frac{\mathbf{r}''_{1,2}(t)}{\left|\mathbf{r}''_{1,2}(t)\right|}, \qquad t \ge t_m + \tau,} \tag{2.3-195}$$

where

$$\boxed{\mathbf{r}''_{1,2}(t) = \mathbf{r}_{1,2} + \Delta t\, \mathbf{V}_0, \qquad t \ge t_m + \tau,} \tag{2.3-196}$$

$\mathbf{r}_{1,2}$ is given by (2.1-14), Δt is given by (2.3-180), and τ is given by (2.3-181). Note that (2.3-196) can be obtained by substituting (2.3-13), (2.3-25), (2.3-75), (2.3-153), (2.3-159), and (2.3-161) into (2.3-28). And by referring to (2.3-91) and (2.3-92), we can write that

$$\boxed{\theta''_{1,2}(t) = \cos^{-1} w''_{1,2}(t), \qquad t \ge t_m + \tau,} \tag{2.3-197}$$

and

$$\boxed{\psi''_{1,2}(t) = \tan^{-1}\left(\frac{v''_{1,2}(t)}{u''_{1,2}(t)}\right), \qquad t \geq t_m + \tau,} \tag{2.3-198}$$

where $u''_{1,2}(t)$, $v''_{1,2}(t)$, and $w''_{1,2}(t)$ are the dimensionless, time-varying direction cosines with respect to the X, Y, and Z axes, respectively, associated with the time-varying unit vector $\hat{n}''_{1,2}(t)$ given by (2.3-195), and τ is given by (2.3-181). Equations (2.3-197) and (2.3-198) are the *angles of scatter* at the receiver that are to be used to evaluate the scattering function of the discrete point scatterer given by (2.3-101).

Finally, let us solve for the time-varying position vector $\mathbf{r}''_2(t)$ from the origin to the receiver. If we substitute (2.3-153) and (2.3-161) into (2.3-95), then

$$\boxed{\mathbf{r}''_2(t) = \mathbf{r}_2 + \Delta t \mathbf{V}_0, \qquad t \geq t_m + \tau,} \tag{2.3-199}$$

where Δt is given by (2.3-180) and τ is given by (2.3-181).

In order to fully appreciate the importance of the results contained in this SAS example, keep in mind that they are based on the *exact time-varying ranges* between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver, and 2) the *exact time-varying angles of incidence* at the discrete point scatterer, and the *exact time-varying angles of scatter* at the receiver that are used to evaluate the frequency dependent scattering function of the discrete point scatterer. In contrast, Bonnifant [8], for example, 1) initially ignores the height of the transmit/receive platform above the ocean bottom when computing ranges, 2) he assumes that the target's reflectivity is constant - not a function of frequency and angles, and 3) that the transmit/receive platform is stationary during signal transmission and reception - the common "stop and hop" assumption that he later attempts to correct for with an approximate phase factor correction. We have made no such assumptions in this example and, as a result, no corrections are necessary.

3 Summary

The complex frequency response of the ocean was derived for the following three different bistatic scattering problems: 1) no motion, 2) only the discrete point scatterer is in motion, and 3) all three platforms (the transmitter, discrete point scatterer, and receiver) are in motion. The propagation of the small-amplitude acoustic signals in the ocean involved in the bistatic scattering problems was treated as transmission through a *linear, time-variant, space-variant filter.*

Scatter from a discrete point scatterer was modeled via the *scattering function*, which is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer, and the direction of wave propagation from the scatterer to the receiver. In addition to the scattering function, frequency-dependent attenuation was taken into account in order to model the propagation of sound from transmitter to discrete point scatterer, and from discrete point scatterer to receiver.

The speed of sound and ambient density of the ocean were treated as *constants*. Therefore, sound rays will travel in straight lines. We only concerned ourselves with solving for the direct ray path between transmitter and discrete point scatterer, and from discrete point scatterer to receiver. As a result, the three platforms were treated as being in an unbounded, homogeneous ocean medium. However, sound propagation between the transmitter and the ocean surface and bottom, and from the ocean surface and bottom to the receiver can be handled in the same way as was done for the discrete point scatterer.

Section 2.1 was devoted to the first bistatic scattering problem, which involves *no* motion - the transmitter, discrete point scatterer, and receiver are *not* in motion. The *exact* solutions for the *angles of incidence* at the discrete point scatterer and the *angles of scatter* at the receiver were derived. An example was worked out at the end of Section 2.1 showing how the general bistatic scattering results reduced for a *monostatic* (*backscatter*) scattering geometry for the no motion case.

Section 2.2 was devoted to the second bistatic scattering problem when only the discrete point scatterer is in motion. Motion was allowed to start at an arbitrary time instant t_m seconds as opposed to zero seconds. Two *new* major results were presented in Section 2.2: 1) the *exact time-varying ranges* between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver were derived, and 2) the *exact time-varying angles of incidence* at the discrete point scatterer, and the *exact time-varying angles of scatter* at the receiver were also derived. An example was worked out at the end of Section 2.2 showing how the general bistatic scattering results reduced for a *monostatic* (*backscatter*) scattering geometry for the case when only the discrete point scatterer is in motion.

Section 2.3 was devoted to the third bistatic scattering problem when all three platforms are in motion. Motion was allowed to start at an arbitrary time instant t_m seconds as opposed to zero seconds. Two *new* major results were presented in Section 2.3: 1) the *exact time-varying ranges* between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver were derived, and 2) the *exact time-varying angles of incidence* at the discrete point scatterer, and the *exact time-varying angles of scatter* at the receiver were also derived. Three examples were worked out at the end of Section 2.3. The first example showed that the exact results derived in Section 2.3 reduced to the exact results derived in Sections 2.1 and 2.2 when appropriate values were used for the various parameters. This is a very important example because it validates the correctness of the general solution derived in Section 2.3. The second example showed how the general bistatic scattering results reduced for a *monostatic* (*backscatter*) scattering geometry for the case when all three platforms are in motion. The third example showed how the general bistatic scattering results can be applied to a synthetic aperture sonar (SAS) system trying to image a nonmoving target on the ocean bottom without having to make several common simplifying assumptions.

Finally, it is important to note that for problems involving motion, the solutions for the exact time-varying ranges between the transmitter and discrete point scatterer, and between the discrete point scatterer and receiver derived in this report are also valid in an *inhomogeneous ocean* where

the speed of sound and ambient density are functions of position since solving for a range represents a problem in mechanics not wave propagation. However, travel times and angles of incidence and scatter are different in an inhomogeneous ocean compared to a homogeneous ocean because of the complicated trajectories of sound rays in an inhomogeneous ocean.

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[3] see [1], pp. 58-69.

[4] see [1], Equations (2.1-17), (2.1-20), and (2.1-22), pp. 6-7.

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